Multiple Hypotheses Mixing Filter for Hybrid Markovian Switching Systems

P. H. R. Q. A. Santana, H. M. Menegaz, G. A. Borges and J. Y. Ishihara

***Abstract*— This work addresses the problem of stochastic state estimation for hybrid Markovian switching systems. The proposed Multiple Hypotheses Mixing Filter (MHMF) combines the Generalized Pseudo Bayes' (GPB) multiple hypotheses tracking with the Interacting Multiple Model's (IMM) estimates mixing in order to improve performance, the later being a particular case of the MHMF. A hypotheses pruning step prevents the filter's output to be degraded by estimates coming from very unlikely hypotheses and the mode transition probabilities are estimated online based on the measurements' likelihoods. A target tracking application shows the MHMF's utility as a stochastic filter for hybrid systems.**

I. INTRODUCTION

HYBRID systems, in a broad sense, denote a class of dynamical systems whose behavior combines continuous and discrete state variables [1], [2]. Because of its versatility, the hybrid modeling framework has been applied to a wide range of applications and extensive work has been done in the field of state estimation for this kind of system. For instance, [3] applies particle filtering for hybrid systems in the context of signal processing. Target tracking, one of the most common applications for hybrid systems, is treated in [4], [5]. In [6], robust Kalman filtering techniques are used for state estimation of hybrid systems with unknown nonlinearities, while [7] performs both robust state estimation and fault diagnosis for uncertain hybrid systems.

Many practical systems have their dynamics described by a set of mathematical models rather than just by one. A switched electronic circuit is one simple example of these such systems, since voltages and currents can undergo sudden changes and have their dynamics altered depending on the switches' logic states. In order to describe these multiple model (MM) systems, a hybrid approach is often appropriate. In this context, the discrete variables usually denote the system's operating mode and define how the continuous state evolves. State estimation for this kind of system generally requires filtering for both the discrete and the continuous state variables and many techniques can be found in [5], [8]–[10] and references therein. Among the most important of them are the Generalized Pseudo Bayes (GPB) [11] and the Interacting Multiple Model (IMM) [10], [12] algorithms, which are based on multiple model Kalman Filters (KFs).

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The GPB filter is a suboptimal approach to the optimal Multiple Hypothesis Tracker (MHT) [11]. After a fixed number of steps, estimates coming from different hypotheses are merged based on their probabilities, rendering the filter possible to be implemented in practice. The IMM algorithm introduced in [12] has a similar structure, but greatly improves performance without increasing the computational load by introducing an estimates mixing step based on the system modes' probability prediction, making this algorithm one of the best choices in terms of cost and efficiency [5]. However, the IMM's hypotheses merging assumptions sometimes become unsuitable when the number of hypotheses being tracked is large or for nonlinear systems where the Gaussian noise approximation is not very adequate. Further, the transition probabilities for the IMM's Markovian mode switching are considered as given parameters.

Besides merging similar hypotheses, each one of them corresponding to a possible path followed by the system's discrete operating mode, works such as [13]–[15] also introduce a pruning step that eliminates estimates coming from hypotheses with considerably low probability. In order to render their algorithms computationally feasible and, at the same time, avoid performance degradation caused by unlikely estimates [9], these works define probability thresholds below which hypotheses are simply discarded. On the other hand, the algorithm presented in [13] differs from many other results in the context of hybrid estimation, for it does not have an estimates merging step for providing more accurate initial conditions for the next iteration of its bank of KFs, relying solely on its hypotheses pruning step for reducing the filter's computational burden. Refer to [16] for a survey on hybrid filtering methods based on multiple hypothesis tracking.

This work's contribution lies in the proposition of a novel multiple hypotheses mixing filter for hybrid Markovian switching systems. Differently from the IMM, which can be considered a particular case of the proposed filter, no restrictions are cast upon the hypotheses merging depth and no previous knowledge on the mode transition probabilities is assumed. Additionally, a hypotheses pruning step prevents the filter's output to be degraded by estimates from very unlikely hypotheses, as discussed in [9]. The filters presented in [14], [15] also perform state estimation for hybrid systems through the tracking of multiple hypotheses. However, these filters assume *a priori* knowledge on the transition probabilities, which can be unrealistic [17]. The multiple hypotheses mixing filter (MHMF) proposed in this work seeks to be an improvement over the existing methods by simultaneously tracking multiple hypotheses and estimating the mode transition probabilities based on a set of noise-corrupted measurements.

This paper is organized as follows. Section II describes the problem of optimal and suboptimal MM filtering and presents the proposed filter's improvements over existing methods. The multiple hypotheses filter's algorithm is presented in Section III and a target tracking application is shown in Section IV. Finally, conclusions are presented in Section V.

II. PROBLEM FORMULATION

For the multiple model filtering problem, consider the discrete time hybrid system

$$x_k = f_{m_k}(x_{k-1}, u_{k-1}, w_{k-1}), \quad (1)$$

$$y_k = h_{m_k}(x_k, v_k), k \in \mathbb{N}, \quad (2)$$

where $x_k \in \mathbb{R}^{n_x}$ is the sampled continuous state vector; $m_k \in \mathbb{M} \triangleq \{1, 2, \ldots, \mathrm{M}\}$ is the system's discrete modal state (mode), which can assume M different values; $f_{m_k} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$ is a possibly nonlinear mode-dependent process evolution function; $h_{m_k} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_y}$ and $y_k \in \mathbb{R}^{n_y}$ are the mode-dependent measurement function and measurement vector, respectively; $u_{k-1} \in \mathbb{R}^{n_u}$ is the input vector; and $v_{k-1} \in \mathbb{R}^{n_v}$ and $w_{k-1} \in \mathbb{R}^{n_w}$ are noise processes. The parameter m_k is assumed to follow a Markov Chain with initial probability vector $p(m_0)$ and transition probability matrix (TPM)

$$\Pi = (\pi_{i,j}), \pi_{i,j} = P\{m_k = j | m_{k-1} = i\}, i, j \in \mathbb{M}.$$

Assuming that neither x_k nor m_k may be directly measurable, one wishes to obtain the joint *a posteriori* probability density function (pdf)

$$p(x_k, m_k | y_{1:k}) = p(x_k | m_k, y_{1:k}) P(m_k | y_{1:k}) \quad (3)$$

based on a sequence $y_{1:k} = \{y_1, y_2, \ldots, y_k\}$ of noise-corrupted measurements generated according to (2). According to (3), it is possible to address the joint estimation of x_k and m_k as two separate problems: estimating the *a posteriori* pdf $p(x_k | m_k, y_{1:k})$ of x_k conditional on the mode; and estimating the discrete conditional modal probability $P(m_k | y_{1:k})$ independently from x_k.

The random mode transitions that occur between samples define the true mode sequence

$$I_k = \{m_1, m_2, \ldots, m_k\} \quad (4)$$

for the Markovian switching system (1)-(2). However, because m_k can only be estimated from the output measurements, (4) cannot be retrieved. Actually, the unknown mode transitions yield the set

$$\Omega_k = \{I_k^{(i)} | i = 1, 2, \ldots, \mathrm{M}^k\} \quad (5)$$

of all possible mode sequences until the k-th sample instant, where $I_k^{(i)}$ denote the i-th hypothesis. As k grows with time, the number of hypotheses in (5) increases exponentially.

Ackerson *et al.* [11] derive the optimal estimator for (1)-(2) in a minimum mean-square error (MMSE) sense in the case where the functions f_{m_k} and h_{m_k} are linear, the TPM is known and the noise processes are Gaussian. In this case, the optimal estimate $\tilde{x}_k$ of the system's state x_k is given by

$$\tilde{x}_k = \sum_{j=1}^{\mathrm{M}^k} \tilde{x}_k^{(j)} P(I_k^{(j)} | y_{1:k}), \quad (6)$$

where $\tilde{x}_k^{(j)} = E\{x_k | I_k^{(j)}, y_{1:k}\}$ is the optimal state estimate given by a Kalman Filter (KF) tracking the hypothesis $I_k^{(j)}$. However, the exponential growth in the number of hypotheses in (6) renders its implementation impossible in practice, since memory and computation requirements are unbounded. Hence, suboptimal approaches using hypotheses management methods, such as merging similar hypotheses and pruning unlikely ones, become necessary in order to implement feasible multiple model (MM) estimators in real-time.

In [8]–[10] and references therein, some filtering algorithms with different strategies for multiple hypotheses handling are presented. Among the most important of them are the Generalized Pseudo Bayes (GPB) [11] and the Interacting Multiple Model (IMM) [10] filters, which handle the hypotheses merging problem differently. The former performs a weighted combination of hypotheses based on their probabilities after some fixed number of steps, whilst the later performs an estimates combination in order to generate new initial conditions for its bank of KFs at each time step. Because of this mixing step, the IMM algorithm exhibits computational requirements comparable with the GPB1, which are linear in the size of the problem (number of modes), whilst its performance is almost the same as that of GPB2, which has quadratic complexity, making this algorithm one of the best choices in terms of cost and efficiency [5]. However, the IMM's fixed depth merging approach can, sometimes, be very restrictive. In order to see that, consider the equations

$$P(m_k{=}i | y_{1:k-1}) = \sum_{j=1}^{M} \pi_{j,i} P(m_{k-1}{=}j | y_{1:k-1}),$$

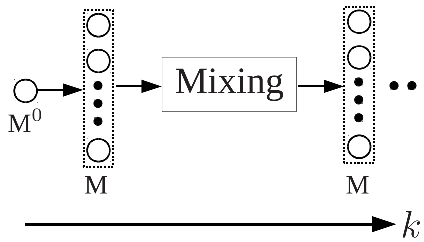
$$p(x_{k-1} | m_k{=}i, y_{1:k-1}) = \frac{\sum_{j=1}^{M} \pi_{j,i} P(m_{k-1}{=}j | y_{1:k-1}) g_{k-1}(j)}{P(m_k{=}i | y_{1:k-1})}, \quad (7)$$

$$g_{k-1}(j) = p(x_{k-1} | m_{k-1} = j, y_{1:k-1}), \quad (8)$$

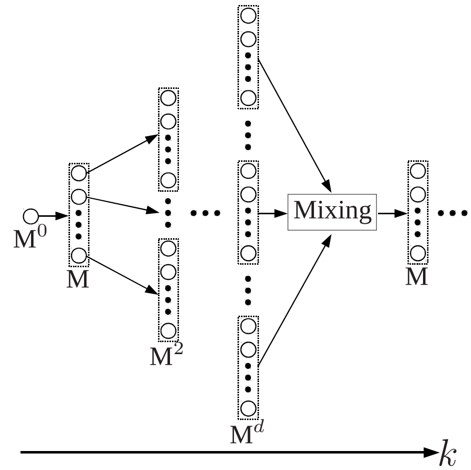
for the IMM's estimates mixing step. According to [10], even if $p(x_0)$ is Gaussian, (8) is, in general, a sum of M^{k-1} weighted Gaussians. Nevertheless, the IMM's fixed depth merging approach depicted in Fig. 1(a) assumes that

$$p(x_{k-1} | m_{k-1} = i, y_{1:k-1}) \sim N(\hat{x}_i(k{-}1), \hat{P}_i(k{-}1)), \quad (9)$$

where $\hat{x}_i(k{-}1)$ and $\hat{P}_i(k{-}1)$ are, respectively, the state estimate and associated error covariance matrix yielded by a KF following the mode $m_{k-1} = i$. Although (9) if often a good approximation for linear hybrid systems with a small number of modes, this may not be the case when the number



(a) IMM's estimates mixing.



(b) MHMF's estimates mixing.

Fig. 1. Estimates mixing with different merging depths.

of hypotheses being tracked is large or for nonlinear systems where the Gaussian noise approximation is not very suitable. Thus, in order to improve the hybrid filter's performance keeping the computational requirements bounded, we improve the IMM's estimates mixing by combining it with the GPB's multiple hypotheses tracking, as shown in Fig. 1(b). Denoting d as the hypotheses merging depth, (8) becomes

$$g_{k-1}(j) = \sum_{i=1}^{\mathrm{M}^d} p(x_{k-1}|m_{k-1}=j, I_{k-1}^{(i)}, y_{1:k-1}) \\ .P(I_{k-1}^{(i)}|m_{k-1}=j, y_{1:k-1}), \tag{10}$$

$$P(I_{k-1}^{(i)}|m_{k-1}{=}j, y_{1:k-1}) = \\ \frac{P(m_{k-1}{=}j|I_{k-1}^{(i)}, y_{1:k-1})P(I_{k-1}^{(i)}|y_{1:k-1})}{P(m_{k-1}{=}j|y_{1:k-1})}, \tag{11}$$

where $I_{k-1}^{(i)}$ is one particular hypothesis being tracked between two estimates mixing. In (11), $P(m_{k-1}{=}j|I_{k-1}^{(i)}, y_{1:k-1})$ is either 0 or 1, depending on which mode corresponds to $I_{k-1}^{(i)}$. It is clear that the IMM's mixing step is a particular case of (10)-(11) with $d = 1$.

III. MULTIPLE HYPOTHESES MIXING FILTER

In the MM filtering problem presented in the previous section, choosing $d > 1$ in (10)-(11) tends to improve the estimates given by (7), but increases the number of hypotheses being calculated. According to [9], having the estimator consider too many models can be as harmful to its performance as considering fewer than necessary. This happens because estimates coming from very unlikely hypotheses end up degrading the filter's overall performance, besides the increase in the computational load. Hence, similarly to [14], [15], a hypotheses pruning step was introduced, which eliminates hypotheses whose probabilities are below a given threshold, avoiding unnecessary calculations and preventing the filter's output degradation.

Furthermore, many MM estimators for Markovian switching systems, including the GPB, the IMM, the MMMH in [14], and the M^{3}H filter in [15], assume previous knowledge on Π, which is rarely the case [17]. Hence, a TPM estimation step based on the system's measurements was incorporated to the filter's algorithm in order to perform the transition probabilities' online estimation. Considering the hybrid model described in (1)-(2), one wishes to obtain

- $\hat{x}_k$, the estimated minimum variance state vector;
- $\hat{p}(m_k)$, the estimated mode probability vector;
- $\hat{\Pi}(k)$, the estimated TPM;

from a sequence of disturbed measurements $y_{1:k}$.

Since the MHMF presented in Section III-B is a generic nonlinear filter, performing a convergence analysis is not possible. In the context of nonlinear systems, this kind of verification is feasible solely by means of Lyapunov function candidates, which are suitable only for specific cases and do not necessarily seek to reduce variances, therefore not being applicable for the MHMF's case itself.

A. *TPM estimation*

Many works, such as [4], [12], [14], [15], [18], [19], concerning state estimation in the context of Markovian jump systems (MJS) assume prior knowledge on the mode transition probabilities, i.e., Π is a given parameter. However, this assumption is usually unrealistic. Choosing an incorrect *a priori* value for Π may degrade the filter's performance and lead to inaccurate values for $\hat{x}_k$ and $\hat{p}(I_k)$, making the online estimation of Π based on $y_{1:k}$ a desirable and important feature.

The algorithm presented in [20] to perform the online estimation of unknown, non stationary TPMs models each row of Π as following a prior Dirichlet distribution and derives a Bayesian mean-variance estimator based on the fact that the Dirichlet distribution is conjugate to the multinomial distribution. However, the estimator [20] assumes perfect mode observation, which is not the case for (1)-(2). For TPM estimation, it has been used the *Quasi-Bayesian* algorithm described in [17] using just the system's measurements as inputs to the TPM estimator. This estimator is incorporated to the hybrid nonlinear filter used to track (1)-(2).

B. Algorithm

As can be seen in (10)-(11) and in Fig. 1(b), perform the state estimation of (1)-(2) consists of tracking multiple hypotheses between two estimates mixing. Moreover, it is also necessary to estimate the hypotheses' probabilities in order to weight the filters' estimates according to how likely their outputs are correct. Combining this multiple hypotheses tracking filter with the TPM estimation algorithm of Section III-A yields a hybrid filter which recursively estimates $\hat{x}_k$, $\hat{p}(m_k)$, and $\hat{\Pi}(k)$.

Multiple Hypotheses Mixing Filter (MHMF) *Let* $\hat{x}_i(k)$ *and* $\hat{P}_i(k)$, $i \in \{1, 2, \ldots, M^q\}$, $q \in \{1, 2, \ldots, d\}$, *be the state vector and associated covariance matrix corresponding to the filter tracking the system hypothesis* $I_k^{(i)}$ *at the* k*-th sample instant, where* d *is the hypotheses merging depth. Let also* y_k *be the system's current output vector and* $0 \leq \varepsilon < 1$ *be the hypotheses pruning threshold. Furthermore, denote* $n(I_k)$ *as the total number of hypotheses at the* k*-th sample instant and* $m_k^{(i)} \in \mathbb{M}$ *as the current mode for hypothesis* $I_k^{(i)}$*. Defining* $\hat{p}_i(I_k) = P(I_k^{(i)}|y_{1:k})$, $\hat{p}_i(m_k) = P(m_k = i|y_{1:k})$, *and assuming initial conditions*

$$\begin{aligned}
\hat{p}(I_0) &= [\hat{p}_1(I_0)\ \hat{p}_2(I_0)\ \ldots\ \hat{p}_M(I_0)],\\
\hat{x}_i(0) &= x(0), x(0) \in \mathbb{R}^{n_x},\\
\hat{P}_i(0) &= P(0), P(0) \in \mathbb{R}^{n_x \times n_x},\\
\hat{\Pi}(0) &= \Pi(0),\\
q(0) &= 1,
\end{aligned}$$

the hybrid data fusion algorithm can be given by the following steps:

i ***Hypotheses probability prediction***

$$\bar{p}_i(I_k) = \hat{\pi}_{a,b}(k-1)\hat{p}_j(I_{k-1}),\ a = m_{k-1}^{(j)}, b = m_k^{(i)},$$

where $i \in \{1, \ldots, n(I_k)\}$, $j \in \{1, \ldots, n(I_{k-1})\}$.

ii ***Hypotheses pruning***

Eliminate hypotheses $I_k^{(i)}$ *with*

$$\frac{\bar{p}_i(I_k)}{\sum_{j=1}^{n(I_k)} \bar{p}_j(I_k)} \leq \varepsilon,$$

normalize the probabilities $\bar{p}_i(I_k)$*, and update* $n(I_k)$ *accordingly.*

iii ***Initial conditions***

- *If* $n(I_k) \leq M$ *or* $q(k) < d$

$$\underline{x}^i(k) = \hat{x}_j(k-1),\ \underline{P}^i(k) = \hat{P}_j(k-1),$$
$$q(k) = q(k-1) + 1,$$

meaning that the hypothesis $I_k^{(i)}$ *was obtained from* $I_{k-1}^{(j)}$.

- *Else*

$$\begin{aligned}
\bar{p}_i(m_k) &= \sum_{j=1}^{n(I_k)} P(m_k = i|I_k^{(j)}, y_{1:k-1})\bar{p}_j(I_k^{(j)}),\\
\underline{x}^i(k-1) &= \sum_{j=1}^{M} \frac{\hat{\pi}_{j,i}(k-1)\hat{p}_j(m_{k-1})r_{k-1}(j)}{\bar{p}_i(m_k)},\\
r_{k-1}(j) &= \sum_{l=1}^{n(I_{k-1})} \hat{x}_l(k-1)\hat{P}(I_{k-1}^{(l)}|m_{k-1} = j, y_{1:k-1}),\\
\underline{P}^i(k-1) &= \sum_{j=1}^{M} \frac{\hat{\pi}_{j,i}(k-1)\hat{p}_j(m_{k-1})\left[\Delta_{k-1}(j) + \delta(i,j)\right]}{\bar{p}_i(m_k)},\\
\Delta_{k-1}(j) &= \sum_{l=1}^{n(I_{k-1})} \hat{P}_l(k-1)\hat{P}(I_{k-1}^{(l)}|m_{k-1} = j, y_{1:k-1}),\\
\delta(i,j) &= \left(r_{k-1}(j) - \underline{x}^i(k-1)\right)(\cdot)^T,\\
q(k) &= 1,
\end{aligned}$$

where $\hat{P}(I_{k-1}^{(l)}|m_{k-1} = j, y_{1:k-1})$ *is calculated according to (11).*

iv ***Filter-dependent prediction step***

$$(\underline{x}^i(k-1), \underline{P}^i(k-1)) \xrightarrow{\text{Prediction}} (\bar{x}_i(k), \bar{P}_i(k)). \tag{12}$$

v ***Filter-dependent correction step***

$$(\bar{x}_i(k), \bar{P}_i(k)) \xrightarrow{\text{Correction}} (\hat{x}_i(k), \hat{P}_i(k)). \tag{13}$$

vi ***Hypothesis probability correction***

$$\begin{aligned}
\hat{p}_i(I_k) &= \frac{p(y_k|I_k^{(i)}, \hat{\Pi}(k-1), y_{1:k-1})\bar{p}_i(I_k)}{c_i},\\
\gamma_p &= \sum_{j=1}^{n(I_k)} \hat{p}_j(I_k),\\
\hat{p}(I_k) &= [\hat{p}_1(I_k)\ \ldots\ \hat{p}_{n(I_k)}(I_k)]^T \left(\frac{1}{\gamma_p}\right).
\end{aligned}$$

vii ***Output generation***

$$\begin{aligned}
\hat{x}_k &= \sum_{i=1}^{n(I_k)} \hat{p}_i(I_k)\hat{x}_i(k),\\
\hat{P}_k &= \sum_{i=1}^{n(I_k)} \hat{p}_i(I_k)\left[\hat{P}_i(k) + (\hat{x}_i(k) - \hat{x}_k)(\cdot)^T\right],\\
\hat{p}_i(m_k) &= \sum_{j=1}^{n(I_k)} P(m_k = i|I_k^{(j)}, y_{1:k})\hat{p}_j(I_k^{(j)}|y_{1:k}),\\
\hat{p}(m_k) &= [\hat{p}_1(m_k)\ \ldots\ \hat{p}_M(m_k)]^T.
\end{aligned}$$

viii ***TPM update****:* $\hat{\Pi}(k-1) \xrightarrow{\text{Algorithm [17]}} \hat{\Pi}(k)$.

No details are given in (12) and (13) because these steps vary depending on the filter chosen to track each system's hypothesis. For example, if (1)-(2) are linear, the KF is a sensible choice. On the other hand, the numerical results presented in Section IV were obtained by using the Extended Kalman Filter (EKF) as the filtering solution. No matter which filter is chosen, step iv takes the initial condition $(\underline{x}^i(k), \underline{P}^i(k))$ for the filter tracking the hypothesis $I_k^{(i)}$ and yields the predicted state and covariance matrix $(\bar{x}_i(k), \bar{P}_i(k))$. Next, step v, based on the system's current output sample y_k, provides the corrected estimates $(\hat{x}_i(k), \hat{P}_i(k))$.

IV. Numerical Results

The target tracking problem is among the most common applications of MM estimation [5]. In fact, the IMM algorithm was motivated by the well-known problem of aircraft tracking by a surveillance radar in Air Traffic Control (ATC) systems [12]. In order to verify the MHMF's performance and compare it with the widely used IMM, a target tracking application for ATC based on an example given in [21] was implemented. The EKF was the filter chosen to track each one of the system's modes, having a computational complexity of $\mathcal{O}(L^3)$ according to [22], where L is the state vector's dimension. Because L is constant for all system's modes, the computational complexity for the MM filters presented in this section becomes a linear function of the number of modes being tracked. Therefore, the computational complexity for both the IMM and the MHMF is $\mathcal{O}(n(I_k))$, where $n(I_k)$ is defined in Section III-B.

Let $x = [p_x\ v_x\ p_y\ v_y\ \Omega]^T$ be the state vector associated with the target tracking application, where p_x and p_y are the Cartesian coordinates along the x and y axes and $v_x = \dot{p}_x$ and $v_y = \dot{p}_y$ are the associated velocities. The term Ω denotes the angular velocity during course changes. For this target tracking example, two distinct dynamic modes are considered. The first one concerns Uniform Motion (UM) and is described by

$$x_k = \begin{bmatrix} 1 & T & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \\ 0 & 0 \end{bmatrix} w_{k-1}, \quad (14)$$

where T denotes the discrete sample period and $w_{k-1} \sim N(0, Q_k)$ is a Gaussian noise process modeling disturbing accelerations. The second dynamic mode addresses course changes through Coordinated Turns (CT) with constant angular velocity Ω, whose model is given by

$$x_k = \begin{bmatrix} 1 & \frac{\sin(\Omega T)}{\Omega} & 0 & -\frac{1-\cos(\Omega T)}{\Omega} & 0 \\ 0 & \cos(\Omega T) & 0 & -\sin(\Omega T) & 0 \\ 0 & \frac{1-\cos(\Omega T)}{\Omega} & 1 & \frac{\sin(\Omega T)}{\Omega} & 0 \\ 0 & \sin(\Omega T) & 0 & \cos(\Omega T) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 & 0 \\ T & 0 & 0 \\ 0 & \frac{1}{2}T^2 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} w_{k-1}. \quad (15)$$

For both the UM and the CT modes, the output model is

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x_k + v_k, \quad (16)$$

$v_k \sim N(0, R_k)$ is uncorrelated to w_{k-1}. Starting from the initial position

$$x_0 = [25000\ -120\ 10000\ 0\ 0]^T,$$

the simulations carried out in this section consider the following trajectory for the aircraft:

1) UM for 30 s;
2) CT with $\Omega = 5\pi/180\ rad/s$ for 7 s;
3) UM for 30 s;
4) CT with $\Omega = -3\pi/180\ rad/s$ for 15 s;
5) UM for 30 s.

First, a merging depth $d = 1$ and a pruning threshold $\varepsilon = 0$ were chosen in order to verify the equivalence between the MHMF and the IMM for this particular case. As expected, the results yielded by the two filters were identical.

TABLE I

AVERAGE RESULTS FOR 100 MONTE CARLO REALIZATIONS.

	d	Avg. hypotheses #	Rel. RMS error
IMM	1	2	1.22
MHMF	2	2.23	1.0
MHMF	3	3.35	0.98

Next, a set of 100 Monte Carlo realizations of the system described by (14)-(16) was used to compare the results for the MHMF and the IMM, which can be seen in Table I, with different parameters. For both situations, a pruning threshold $\varepsilon = 0.02$ was used to eliminate unlikely hypotheses. In order to show the benefits of performing the TPM's online estimation, all filters started with a poor initial estimate

$$\Pi(0) = \begin{bmatrix} 0.2 & 0.8 \\ 0.1 & 0.9 \end{bmatrix}.$$

As shown in Table I, the MHMF performed better than the IMM in a RMS error sense in both cases. The initial uncertainty about the system's parameters was responsible for a slight increase in the computational load, since a greater number of hypotheses was necessary in order to correctly track the system's state. However, the relative RMS errors confirm that the tracking performance gains were substantial and Fig. 2 shows the estimation errors for one of the Monte Carlo realizations with $d = 2$. Although the MHMF performed better for $d = 3$, the considerable increase in the average number of tracked hypotheses shows that considering a greater number of hypotheses is not always better, which goes in agreement with the results in [9]. Furthermore, it is important to stress the fact that the MHMF outperformed the IMM in all the considered simulations.

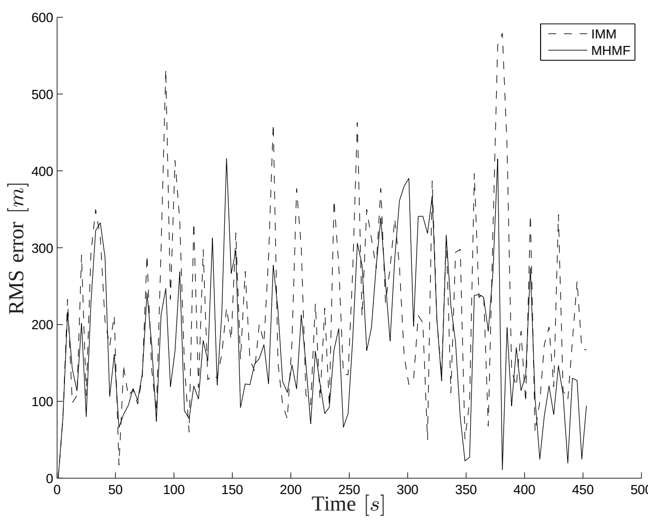


Fig. 2. RMS position estimation errors.

In a second analysis, the MHMF's tolerance to modeling errors was investigated. For this situation, the constant TPM

$$\Pi = \begin{bmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{bmatrix}$$

TABLE II

COMPARATIVE RESULTS FOR THE ATC SYSTEM WITH MODELING ERRORS.

d	Avg. hypotheses #	(MHMF / IMM) RMS error ratio
2	3.0	0.35
3	4.7	0.32
4	7.5	0.26

was used for both the IMM and the MHMF. Instead of (14), the aircraft's UM was modeled by

$$x_k = \begin{bmatrix} 1 & T & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \\ 0 & 0 \end{bmatrix} w_{k-1}, \quad (17)$$

where an additional 1 was introduced in the last line of the process evolution matrix in order to simulate an incorrect angular velocity effect. The tracking results for this situation considered no hypotheses pruning ($\varepsilon = 0$).

One can conclude from the results shown in Table II that considering a larger number of hypotheses improves the filter's tracking performance. As exposed in Section II, the little number of hypotheses considered by the IMM can, sometimes, be a very restrictive assumption, making the filter sensitive to modeling errors. The RMS error ratios in Table II between the MHMF and the IMM clearly show a filtering improvement as the merging depth grows. For instance, the MHMF's RMS error is approximately 0.26 times the IMM's error for $d = 4$. Hence, if increasing the computational load is possible, it would be interesting to consider a larger merging depth to mitigate problems originated from modeling errors.

V. CONCLUSIONS

This work proposed a novel multiple hypothesis mixing filter for hybrid Markovian switching systems. The MM filtering problem was described in Section II and some of the existing methods' limitations were addressed. The MHMF described in Section III is able to simultaneously track multiple hypotheses, perform an estimates mixing similarly to the IMM, and estimate the mode transition probabilities based on the measurements' likelihoods. The numerical results presented in Section IV indicate the MHMF's advantages over the classical IMM, which is a particular case of the proposed filter. The MHMF's variable merging depth and pruning threshold offer the possibility of adjusting the trade-off between computational load and tracking performance, rendering the filter suitable to a wide range of applications, including systems liable to modeling errors.

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