Scaled Minimum Unscented Multiple Hypotheses Mixing Filter

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***Abstract*— This works brings two new contributions. First, it introduces the Scaled Minimum Unscented Multiple Hypotheses Mixing Filter, a novel filter for hybrid dynamical systems that 1) uses a new minimum set of sigma points along with the scaled unscented transform in a hybrid framework; 2) can estimate the Markovian Transition Probability Matrix in real-time; 3) features a pruning step that reduces the filter's computational effort and prevents its estimates from being degraded by very unlikely hypotheses; and 4) has a mixing step with merging depth greater than one. Second, we present a result revealing the conservativeness of one of the scaled unscented transform forms.**

I. Introduction

HYBRID systems are, in a wide sense, a class of dynamical systems whose behavior combines both continuous and discrete state variables [1]. This is clearly a very general definition, giving researchers leeway to model several different problems of interest using the hybrid systems framework. In the context of this paper, we choose our discrete variables to denote the system's operating mode, where by "mode" we mean a set of differential or difference equations that govern the evolution of the continuous portion of the state vector. Such an approach bears a "divide and conquer" intuition, where we implicitly assume that complex nonlinear dynamics can often be better described in terms of a set of interacting simpler models, as opposed to a single, very complex nonlinear model that might not even be tractable. To give a grounded example, imagine a switched electronic circuit as our candidate hybrid system, where the state of all logical switches defines a set of discrete mode values that can abruptly modify the way currents and voltages vary throughout the circuit's connections. On the other hand, framing your problem as a hybrid system introduces the challenge of determining the correct system mode at all times, since this is generally a hidden quantity from our state estimator.

State estimation for hybrid systems has been a topic of great scientific interest in recent years, as shown by the large number and variety of papers. For instance, [2] applies particle filtering for hybrid systems in the context of signal processing. Target tracking, one of the most common applications for hybrid systems, is treated in [3]. In [4], robust Kalman filtering techniques are used for state estimation of hybrid systems with unknown nonlinearities, while [5] performs both robust state estimation and fault diagnosis for uncertain hybrid systems. Hybrid state estimation generally requires filtering for both the discrete and the continuous state variables and many techniques can be found in [6], [7] and references therein. Among them, it is important to mention the Generalized Pseudo Bayes (GPB) [8] and the Interacting Multiple Model (IMM) [9], [10] algorithms, which are based on running a bank of Kalman Filters (KFs) simultaneously to keep track of several different discrete modes with Markovian transitions between them.

The GPB filter is a suboptimal approach to the optimal Multiple Hypothesis Tracker (MHT) [8]. After a fixed number of steps, estimates coming from different hypotheses are merged based on their probabilities, rendering the filter possible to be implemented in practice. The IMM algorithm introduced in [9] has a similar structure, but greatly improves performance without increasing the computational load by introducing an estimates mixing step based on the prediction of system modes' probabilities, making this algorithm one of the best choices in terms of computational cost and efficiency [7]. However, the IMM's hypotheses merging assumptions sometimes become unsuitable when the number of hypotheses being tracked is large or for nonlinear systems where Gaussian noise approximations are not very adequate.

The Multiple Hypotheses Mixing Filter (MHMF) - of which the IMM is a particular case - recently introduced by [11] showed itself to be a good alternative for hybrid estimation, since it generalizes the IMM's hypotheses merging step and requires no previous knowledge about the Markovian mode transition probabilities. In addition, a hypotheses pruning step prevents the MHMF's output to be degraded by estimates from very unlikely hypotheses, a phenomenon discussed in more detail in [12]. In order to deal with nonlinear models, the authors in [11] used the Extended Kalman Filter (EKF) in their original formulation. Nevertheless, this work shows that combining the MHMF with the Unscented Kalman Filter (UKF) [13] yields a better performing hybrid filter, which is coherent with prior work that demonstrates that the UKF outperforms the EKF in many applications [14]–[16].

The UKF is a nonlinear filter that uses the Unscented Transform (UT) [13] to approximate the probability density function (PDF) of random variables (RV) using a set of deterministic weighted samples called *sigma points*. There are several different variations of the UT, such as the symmetric [13], the reduced [17], the reduced spherical [18], the minimum [19], and the scaled [20]. In this work, we introduce the Scaled Minimum Unscented Multiple Hypothesis Mixing Filter (ScMinUMHMF), an improvement over the original

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MHMF that combines the minimum and scaled Unscented Transforms in order to better propagate uncertainty information while keeping the filter's computational complexity to a minimum[1].

Our last contribution in this work is to demonstrate that the two versions of the scaled UT first presented in [20] are not equivalent, which does not agree with the authors' original claim. In fact, we show that the second form, which happens to be the most used[2], is more conservative than the first one.

This work is organized as follows. Section II defines the multiple model state estimation problem, while Section III proves that the two versions of the scaled UT originally introduced in [20] are actually distinct. Then, Section IV shows how the scaled and minimum Unscented Transforms are combined with the MHMF filter to create the ScMinUMHMF. Finally, simulation results in Section V show how our proposed filter outperforms the original formulation of the MHMF, followed by our conclusions in Section VI.

II. Problem Statement

A multiple model (MM) system, the particular case of hybrid systems addressed in this paper, can be described as

$$\begin{aligned} x_k &= f_{m_k}(x_{k-1}, u_{k-1}, w_{k-1}), \\ y_k &= h_{m_k}(x_k, v_k), k \in \mathbb{N}, \end{aligned} \tag{1}$$

where k is the time step; $x_k \in \mathbb{R}^{n_x}$ is the continuous state vector; $m_k \in \mathbb{M} \triangleq \{1, 2, \ldots, \mathrm{M}\}$ is the system's discrete modal state (mode), which can assume M different values; $f_{m_k}:\mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_x}$ is a possibly nonlinear mode-dependent process evolution function; $h_{m_k}:\mathbb{R}^{n_r} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_y}$ and $y_k \in \mathbb{R}^{n_y}$ are the mode-dependent measurement function and measurement vector, respectively; $u_{k-1} \in \mathbb{R}^{n_u}$ is the input vector; and $v_{k-1} \in \mathbb{R}^{n_v}$ and $w_{k-1} \in \mathbb{R}^{n_w}$ are noise processes. The parameter m_k is assumed to follow a Markov Chain with initial probability vector $p(m_0)$ and time-invariant transition probability matrix (TPM)

$$\Pi = (\pi_{i,j}), \pi_{i,j} = P\{m_k = j | m_{k-1} = i\}, i, j \in \mathbb{M}.$$

Given models of the form (1) for every possible value of m_k and a sequence of measurements $y_{1:k} = \{y_1, y_2, \ldots, y_k\}$, the MM filtering problem consists of determining the following quantities:

- the minimum variance estimate of the continuous state vector $\hat{x}_k$, along with its covariance matrix $\hat{P}_k$;
- the estimated discrete mode probability vector $\hat{p}(m_k)$;
- the estimated TPM $\hat{\Pi}(k)$.

The ScMinUMHMF proposed in Section IV is an algorithm that solves the MM filtering problem by combining the MHMF in [11] with the scaled and minimum Unscented Transforms introduced in [20] and [19], respectively, and further explained in the next section.

[1]The UKF's computational cost is proportional to the number of *sigma points* [13].

[2]We could not find an application using the first scaled form.

III. Scaled Unscented Transform

Consider the random vector $X \in \Re^n$ and the function $f:\Re^n \to \Re^m$, which defines the random variable $Y := f(X)$. The Unscented Transform (UT) approximates the probability density function of X, the prior random variable, by means of a set of deterministic samples called *sigma points*, which correspond to a discrete probability mass function approximation. One important advantage of this nonlinear uncertainty propagation technique is that it does not require the calculation of Jacobian or Hessian matrices of nonlinear functions, which are necessary whenever linearization-based methods are used [13]. Moreover, if the *sigma points'* central moments are equal to the central moments of X up to the $2k$-th order inclusively, the Taylor Series for the sample mean and covariance of the transformed *sigma points* will be equal, respectively, to the Taylor Series of the mean and covariance matrix of Y up to the $2k$-th order, inclusively [13].

The Scaled Unscented Transform [20], a variation of the original UT, has the property that the i-th term in the Taylor Series expansion of the transformed *sigma points'* mean and (modified) covariance matrix is scaled by an arbitrary term α^{i-2}, where $i > 2$, without incurring any increase in computational cost. The author in [20] presents this transform in two supposedly equivalent ways: i) as a function transforming directly the prior distribution of X; and ii) as a function transforming a previous sigma set. Although ii) has become the most used method, we now show that it is, in fact, more conservative than i).

Method ii) for calculating the Scaled UT consists of obtaining the scaled sigma points, χ_i', and weights, w_i', from a previous set of sigma points, χ_i, and weights, w_i, according to the equations

$$\chi_i' = \chi_1 + \alpha\left(\chi_i - \chi_1\right),$$

$$w_i' = \begin{cases} \frac{w_1}{\alpha^2} + 1 - \frac{1}{\alpha^2} & , i = 1, \\ \frac{w_i}{\alpha^2} & , i = 2, ..., N, \end{cases}$$

where α is a positive scaling parameter [20]. The author in [20] claims that the sample mean and covariance matrix of the scaled set of *sigma points* are equal to the mean and covariance matrix of X, respectively, for any previous sigma set. However, consider the following counterexample.

Example 1 *Let be the random vector $X \in \Re^2$ with mean $\bar{X} = [0\ 0]^T$ and covariance matrix $P_{XX} = I_2$ and let be the sigma points and weights*

$$\chi_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \ \chi_2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \ \ \chi_3 = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix},$$

$$\chi_2 = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}, \quad w_1 = w_2 = w_3 = w_4 = \frac{1}{4}.$$

For $\alpha = 0.5$ and $\chi_1' = \chi_1 = [0\ 2]^T$, the scaled sigma points and weights become

$$\chi_1' = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \chi_2' = \begin{bmatrix} 0,5\sqrt{2} \\ 0,5\sqrt{2} \end{bmatrix}, \chi_3' = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\chi_4' = \begin{bmatrix} 0,5\sqrt{2} \\ -0,5\sqrt{2} \end{bmatrix}, \quad w_1 = -2; \quad w_2 = w_3 = w_4 = 1.$$

Therefore, the sample mean, $\mu_{\chi'}$, and the sample covariance matrix, $\Sigma_{\chi'}$, of the scaled sigma set are

$$\mu_{\chi'} = \sum_{i=1}^{4} w_i \chi_i = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\Sigma_{\chi'} = \sum_{i=1}^{4} w_i \left(\chi_i' - \mu_{\chi'}\right)\left(\chi_i' - \mu_{\chi'}\right)^T = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \neq I_2.$$

The example shows that the sample mean and the sample covariance matrix of χ' are not equal to the mean and covariance matrix of X respectively. In fact, as one can see in the following theorem, this property is only guaranteed for scaling method ii) in case one of the *sigma points* is equal to the mean of X.

Theorem 1 *Let be $X \sim (\bar{X}, P_{XX})$ and the function $f : \Re^n \mapsto \Re^m$ that defines Y, $Y := f(X)$ the set of sigma points $\{\chi_i, w_i | w_i \in \Re, \chi_i \in \Re^n; i = 1, 2, ..., N\}$. Let also be the set of scaled sigma points $\left\{\chi_i^{'}, w_i^{'}\right\}$. Then, the following statements are true:*

1) $\sum_{i=1}^{N} w_i = 1 \Rightarrow \sum_{i=1}^{N} w'_i = 1.$
2) $\mu_{\chi'} = \frac{1}{\alpha}\mu_\chi + \chi_1\left(1 - \frac{1}{\alpha}\right).$
3) $\chi_1 = \mu_\chi \Rightarrow \mu_{\chi'} = \mu_\chi.$
4) *If* $\alpha \neq 1$, *then* $\chi_1 = \mu_\chi \Leftrightarrow \mu_{\chi'} = \mu_\chi.$
5) $\chi_1 = \mu_\chi \Rightarrow \Sigma_{\chi'\chi'} = \Sigma_{\chi\chi}.$

Proof: Suppose $\sum_{i=1}^{N} w_i = 1$, then

$$\sum_{i=1}^{N} w'_i = \left(\frac{w_1}{\alpha^2} + 1 - \frac{1}{\alpha^2}\right)\chi_1 + \sum_{i=2}^{N} \frac{w_i}{\alpha^2}\left(\chi_1 + \alpha\left(\chi_i - \chi_1\right)\right)$$
$$= \frac{1}{\alpha^2}\left(-1 + \alpha^2 + 1\right) = 1,$$

which proves assertive 1. For the second assertive, we have

$$\begin{aligned} \mu_{\chi'} &:= \sum_{i=1}^{N} w'_i \chi_i' \\ &= \left(\frac{w_1}{\alpha^2} + 1 - \frac{1}{\alpha^2}\right)\chi_1 + \sum_{i=2}^{N} \frac{w_i}{\alpha^2}\left(\chi_1 + \alpha\left(\chi_i - \chi_1\right)\right) \\ &= \frac{1}{\alpha}\sum_{i=1}^{N} w_i \chi_i + \chi_1\left(1 - \frac{1}{\alpha}\right) \\ &= \frac{1}{\alpha}\mu_\chi + \chi_1\left(1 - \frac{1}{\alpha}\right), \end{aligned} \tag{2}$$

which proves assertive 2. Now suppose $\chi_1 = \mu_\chi$, then, from (2), $\mu_{\chi'} = \mu_\chi$ which proves assertive 3 and one part of assertive 4. On the contrary, suppose $\alpha \neq 1$ and $\mu_{\chi'} = \mu_\chi$, then, from (2), $\chi_1 = \mu$, which completes the proof for assertive 4. Finally, suppose $\chi_1 = \mu_\chi$, then,

$$\begin{aligned} \Sigma_{\chi'\chi'} &:= \sum_{i=1}^{N} w'_i \left(\chi_i' - \mu_{\chi'}\right)\left(\chi_i' - \mu_{\chi'}\right)^T \\ &= \sum_{i=1}^{N} \frac{w_i}{\alpha^2}\left(\alpha\left(\chi_i - \mu_\chi\right)\right)\left(\alpha\left(\chi_i - \mu_\chi\right)\right)^T \\ &= \Sigma_{\chi\chi}, \end{aligned}$$

which proves the last assertive. ■

On the other hand, form i) of the scaled unscented transform has no restrictions. In the next section, we combine form i) of the Scaled UT and the Minimum UT [19] with the MHMF in [11] and introduce the Scaled Minimum Unscented Multiple Hypothesis Mixing Filter.

IV. Scaled Minimum Unscented Multiple Hypothesis Mixing Filter

This section presents the Scaled Minimum Unscented Multiple Hypothesis Mixing Filter (ScMinUMHMF), a novel filter for multiple model systems described as (1) that combines positives features from the previously introduced MHMF with the uncertainty propagation advantages of the Scaled and Minimum variants of the Unscented Transform. The latter improve the ScMinUMHMF's accuracy when propagating uncertainty through nonlinear transformations, while the features inherited from the MHMF give the ScMinUMHMF the following properties:

- Variable hypothesis merging depth, where by hypothesis we mean a particular branch of the discrete mode m_k evolution tree. For further details, please refer to [11];
- A hypothesis pruning step that reduces the filter's computational effort and also prevents its estimates from being degraded by very unlikely state state estimates (see [12]);
- Online estimation of Π, the Markovian Transition Probability Matrix (TPM) for m_k that is assumed to be time-invariant. In fact, many MM estimators for Markovian switching systems, including the GPB, the IMM, and the modern M^3H filter in [21], assume previous knowledge about the TPM, which is rarely the case [22]. This TPM estimation step is performed by the *Quasi-Bayesian* algorithm described in [22] using just the system's measurements as inputs.

There are also several reasons behind our choice to use the Unscented Transform to propagate uncertainties in our filter. First, the scientific literature has many examples of applications where Unscented Kalman Filters performed better when compared to Extended Kalman Filters,e.g., [13]–[16]. Second, for the MM estimation problem defined in (1), Minimum Unscented Kalman Filters, which are the lightest UKFs to compute, are a very suitable choice due to the fact that such MM estimation problems can easily require great computational effort. Finally, the Scaled Unscented Kalman Filter, for the reasons already stated on section III, can be viewed as an improvement over an UKF using the original form of the UT. However, since the Minimum Unscented Filter used in the ScMinUMHMF formulation has no point equal to the prior random variable's mean, the ScMinUMHMF can feature the first form of the Scaled UT, as formally proved in the previous section.

It is useful to define the following:

Notation 1 *The followings are notations used in this work:*

- $\sqrt{A}$ *stands for a square-root matrix of the matrix* A *such that* $A = \sqrt{A}\sqrt{A}^T$.

- $[A]_{ij}$ *stands for the* i*-th row and* j*-th column element of the matrix* A*.* $[A]_{*j}$ *stands for the* j*-th column of* A *and* $[A]_{i*}$ *stands for the* i*-th row of* A*,*

Following the same notation as [11], we denote by $\hat{x}_i(k)$ and $\hat{P}_i(k)$, $i \in \{1,2,\ldots,M^q\}$, $q \in \{1,2,\ldots,d\}$, the state vector and associated covariance matrix corresponding to the filter tracking the system hypothesis[3] $I_k^{(i)}$ at the k-th sample instant, where d is the hypotheses merging depth. Let also y_k be the system's current output vector and $0 \le \varepsilon < 1$ be the hypotheses pruning threshold. Furthermore, denote $n(I_k)$ as the total number of hypotheses at the k-th sample instant and $m_k^{(i)} \in \mathbb{M}$ as the current mode for hypothesis $I_k^{(i)}$. Defining $\hat{p}_i(I_k){=}P(I_k^{(i)}|y_{1:k})$, $\hat{p}_i(m_k){=}P(m_k{=}i|y_{1:k})$, and assuming initial conditions

$$\begin{aligned}
\hat{p}(I_0) &= [\hat{p}_1(I_0)\ \hat{p}_2(I_0)\ \ldots\ \hat{p}_M(I_0)],\\
\hat{x}_i(0) &= x(0), x(0) \in \mathbb{R}^{n_x},\\
\hat{P}_i(0) &= P(0), P(0) \in \mathbb{R}^{n_x \times n_x},\\
\hat{\Pi}(0) &= \Pi(0), \quad q(0) = 1,
\end{aligned}$$

the algorithm for the ScMinUMHMF's can be stated as follows.

ScMinUMHMF *Scaled Minimum Unscented Multiple Hypotheses Mixing Filter's algorithm consists of the following steps:*

i ***Hypotheses probability prediction***

$$\bar{p}_i(I_k) = \hat{\pi}_{a,b}(k{-}1)\hat{p}_j(I_{k-1}),\ a = m_{k-1}^{(j)}, b = m_k^{(i)},$$

where $i \in \{1,\ldots,n(I_k)\}$, $j \in \{1,\ldots,n(I_{k-1})\}$.

ii ***Hypotheses pruning***

Eliminate hypotheses $I_k^{(i)}$ *with*

$$\frac{\bar{p}_i(I_k)}{\sum_{j=1}^{n(I_k)} \bar{p}_j(I_k)} < \varepsilon,$$

renormalize the probabilities $\bar{p}_i(I_k)$*, and update* $n(I_k)$ *accordingly.*

iii ***Initial conditions***

- *If* $q(k) < d$

$$\begin{aligned}
\underline{x}^i(k) &= \hat{x}_j(k{-}1),\ \underline{P}^i(k){=}\hat{P}_j(k{-}1),\\
q(k) &= q(k{-}1) + 1,
\end{aligned}$$

meaning that the hypothesis $I_k^{(i)}$ *was obtained from* $I_{k-1}^{(j)}$.

- *Else*

$$\hat{p}_i(m_{k-1}){=}\sum_{j=1}^{n(I_{k-1})} \hat{p}_j(I_{k-1}|m_{k-1}{=}i),$$

$$\bar{p}_i(m_k){=}\sum_{j=1}^{n(I_k)} \bar{p}_j(I_k|m_k{=}i),$$

$$\underline{x}^i(k{-}1){=}\sum_{j=1}^{M} \frac{\hat{\pi}_{j,i}(k{-}1)\hat{p}_j(m_{k-1})r_{k-1}(j)}{\bar{p}_i(m_k)},$$

$$r_{k-1}(j){=}\sum_{l=1}^{n(I_{k-1})} \hat{x}_l(k{-}1)\hat{P}(I_{k-1}^{(l)}|m_{k-1}{=}j, y_{1:k-1}),$$

$$\underline{P}^i(k{-}1){=}\sum_{j=1}^{M} \frac{\hat{\pi}_{j,i}(k{-}1)\hat{p}_j(m_{k-1})\left[\Delta_{k-1}(j){+}\delta(i,j)\right]}{\bar{p}_i(m_k)},$$

$$\Delta_{k-1}(j){=}\sum_{l=1}^{n(I_{k-1})} \hat{P}_l(k{-}1)\hat{P}(I_{k-1}^{(l)}|m_{k-1}{=}j, y_{1:k-1}),$$

$$\delta(i,j){=}\left(r_{k-1}(j){-}\underline{x}^i(k{-}1)\right)(\cdot)^T, \quad q(k){=}1,$$

where $\hat{P}(I_{k-1}^{(l)}|m_{k-1}{=}j, y_{1:k-1})$ *is calculated according to*

$$P(I_{k-1}^{(i)}|m_{k-1}{=}j, y_{1:k-1}) = \frac{P(m_{k-1}{=}j|I_{k-1}^{(i)}, y_{1:k-1})P(I_{k-1}^{(i)}|y_{1:k-1})}{P(m_{k-1}{=}j|y_{1:k-1})}, \tag{3}$$

iv ***Filter-dependent prediction step***

1) *choose* $v := [v_1 \ldots v_n]^T \in \Re^n$, $v_i \neq 0$*, for* $i = 1,\ldots,n$
2) *Sigma points and its weights:*

$$w_0 = \frac{1}{1 + \sum_{i=1}^n v_i^2},\ w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} := w_0 \begin{bmatrix} v_1^2 \\ \vdots \\ v_n^2 \end{bmatrix},$$

$$E = \sqrt{\underline{P}^i(k{-}1)}\left(I + vv^T\right)^{-\frac{1}{2}} \frac{1}{\sqrt{w_0}} \begin{bmatrix} \frac{1}{v_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{v_n} \end{bmatrix},$$

$$e := -\frac{1}{w_0}Ew,$$

$$\left[\underline{\chi}_k^0 \ \cdots \ \underline{\chi}_k^n\right] := \left[\ e \quad E\ \right] + [\underline{x}_i(k)]_{1\times(n+1)}.$$

3) *Choose a scaling factor* $0 \le \alpha \le 1$.
4) *Predicted sigma points:*

$$\bar{\chi}_k^j = \frac{f_{m_k}\left(\underline{x}_i(k) + \alpha[\underline{\chi}_k^j - \underline{x}_i(k)]\right) - f_{m_k}(\underline{x}_i(k))}{\alpha^2} + f_{m_k}(\underline{x}_i(k)).$$

5) *Predicted mean and covariance matrix:*

$$\bar{x}_i(k) = \sum_{i=1}^{n+1} w_i \bar{\chi}_k^j,$$

$$\bar{P}_i(k) = \sum_{i=1}^{n+1} w_i \left(\bar{\chi}_k^j - \bar{x}_i(k)\right)\left(\bar{\chi}_k^j - \bar{x}_i(k)\right)^T + Q_k.$$

v ***Filter-dependent correction step***

1) *Measurement's predicted sigma points:*

$$\gamma_k^j = \frac{h_{m_k}\left(\bar{x}_i(k) + \alpha[\bar{\chi}_k^j - \bar{x}_i(k)]\right) - h_{m_k}(\bar{x}_i(k))}{\alpha^2}$$

[3] Please refer to [11] for a very detailed explanation of system hypotheses.

$$+ h_{m_k}(\bar{x}_i(k)).$$

2) *Measurement's predicted mean and covariance matrix:*

$$\bar{y}_k = \sum_{i=1}^{n+1} w_i \gamma_k^j,$$
$$\bar{P}_i^y(k) = \sum_{i=1}^{n+1} w_i \left(\gamma_k^j - \bar{y}_k\right)\left(\gamma_k^j - \bar{y}_k\right)^T + R_k.$$

3) *Cross-covariance matrix:*

$$C_i(k) = \sum_{i=1}^{n+1} w_i \left(\bar{\chi}_k^j - \bar{x}(k)\right)\left(\gamma_k^j - \bar{y}_k\right)^T.$$

4) *Correction of the predicted estimates:*

$$G_k = C_i(k)\left(\bar{P}_i^y(k)\right)^{-1},$$
$$\hat{x}_i(k) = \bar{x}_i(k) + G_k\left(y_k - \bar{y}_k\right),$$
$$\hat{P}_i(k) = \bar{P}_i(k) - G_k \bar{P}_i^y(k) G_k^T.$$

vi ***Hypothesis probability correction***

$$\hat{p}_i(I_k) = \frac{p(y_k|I_k^{(i)}, \hat{\Pi}(k-1), y_{1:k-1})\bar{p}_i(I_k)}{c_i},$$
$$\gamma_p = \sum_{j=1}^{n(I_k)} \hat{p}_j(I_k),$$
$$\hat{p}(I_k) = [\hat{p}_1(I_k) \ \ldots \ \hat{p}_{n(I_k)}(I_k)]^T \left(\frac{1}{\gamma_p}\right).$$

vii ***Output generation***

$$\hat{x}_k = \sum_{i=1}^{M} \hat{p}_i(I_k)\hat{x}_i(k),$$
$$\hat{P}_k = \sum_{i=1}^{M} \hat{p}_i(I_k)\left[\hat{P}_i(k) + (\hat{x}_i(k) - \hat{x}_k)(\cdot)^T\right],$$
$$\hat{p}_i(m_k) = \sum_{j=1}^{n(I_k)} \hat{p}_j(I_k|m_k{=}i),$$
$$\hat{p}(m_k) = [\hat{p}_1(m_k) \ \ldots \ \hat{p}_M(m_k)]^T.$$

viii ***TPM update***: $\hat{\Pi}(k-1) \xrightarrow{\textit{Algorithm [22]}} \hat{\Pi}(k)$.

V. Simulation

In this section, we perform a relative performance evaluation between the ScMinUMHMF's and the MHMF using the same simulated target tracking application presented in [11].

The state vector is $x = [p_x \ v_x \ p_y \ v_y \ \Omega]^T$, where p_x and p_y are respectively the Cartesian coordinates along the x and y axes, v_x and v_y are the associated velocities, and Ω is the angular velocity during course changes. Two different dynamical models are used, namely Uniform Motion (UM) and Coordinated Turns (CT) (see [23]):

- Uniform Motion (UM):

$$x_k = \begin{bmatrix} 1 & T & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \\ 0 & 0 \end{bmatrix} q_k^{um},$$

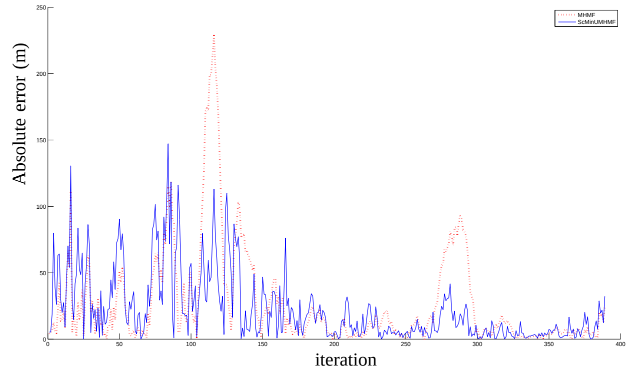

Fig. 1. Absolute error on the x coordinate in each iteration.

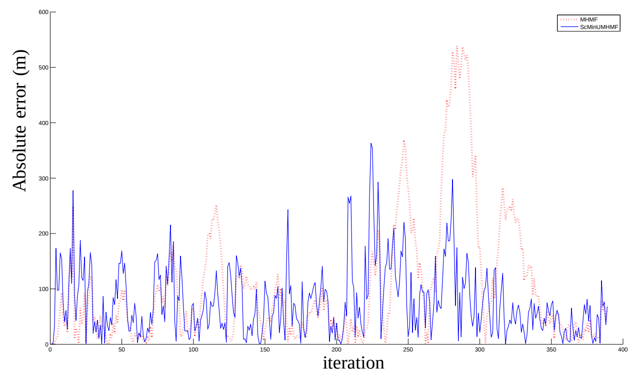

Fig. 2. Absolute error on the y coordinate in each iteration.

- Coordinated Turns (CT):

$$x_k = \begin{bmatrix} 1 & \frac{\sin(\Omega T)}{\Omega} & 0 & -\frac{1-\cos(\Omega T)}{\Omega} & 0 \\ 0 & \cos(\Omega T) & 0 & -\sin(\Omega T) & 0 \\ 0 & \frac{1-\cos(\Omega T)}{\Omega} & 1 & \frac{\sin(\Omega T)}{\Omega} & 0 \\ 0 & \sin(\Omega T) & 0 & \cos(\Omega T) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 & 0 \\ T & 0 & 0 \\ 0 & \frac{1}{2}T^2 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} q_k^{ct}. \quad (4)$$

where T denotes the discrete sample period and $q_k^{um} \sim N(0, Q^{um})$ and $q_k^{um} \sim N(0, Q^{um})$ are the Gaussian noise processes.

The output model is, for both the UM and the CT modes,

$$y_k = \begin{bmatrix} \sqrt{p_x^2 + p_y^2} \\ \arctan(\frac{p_y}{p_x}) \end{bmatrix} x_k + r_k,$$

$r_k \sim N(0, R)$ is uncorrelated to q_k.

Figures 1 and 2 and Table I bring the simulated results. Due to its improved uncertainty propagation, the ScMinUMHMF was able to better match the estimated system mode, which in turn determines the model for temporal evolution of the continuous state vector x, with the actual system mode in the simulation, explaining why its estimation errors are considerably smaller when compared to the MHMF.

TABLE I

RMS ERROR FOR THE ENTIRE TRAJECTORY

MHMF	ScMinUMHMF
13.07	3.576

VI. CONCLUSION

This work presented two new contributions. First, we introduced the Scaled Minimum Unscented Multiple Hypotheses Mixing Filter, a novel stochastic filter in the context of multiple model systems (Algorithm 1). Second, we demonstrated that the two forms of the Scaled Unscented Transform presented in [20] are not equivalent and selected the most appropriate version to be incorporated into the ScMinUMHMF's equations. This new filter combines the benefits of the state-of-the-art Multiple Hypotheses Mixing Filter with the Scaled and Minimum Unscented Transforms. Simulated results using a target tracking application showed that our new filter outperforms the original MHMF using a bank of Extended Kalman Filters for state estimation.

Concerning the second result of this paper, we showed that most used form of the Scaled Unscented Transform is actually more conservative than the alternative formulation. In particular, we showed that the common Scaled Unscented Transform only works when the previous non-scaled sigma set has a sigma point equal to the prior RV's mean (Theorem 1).

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