Robust Stability of Networked Control Systems

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6 Acknowledgments
Discrete Control System
Networked Control System

![Networked Control System Diagram]
Introduction

Networked Control Systems

Applications
- Control over sensor networks;
- Remote surgery;
- Automated highway systems;
- Unmanned aerial vehicles.

Network-Induced Delay
- Time-varying;

Packet Loss
- Retransmitted or discarded;

According to Zhang et al. (2001), the networked delay and the packet dropout can degrade the performance of control systems and can even cause destabilization.
1. Closed-loop NCS with the possibility of dropping data and disordering;

2. Single packet transmission: all data lumped into one network packet;

3. Sensor module is clock-driven with sampling period $h$;

4. Controller and actuator modules are event-driven;

5. Actuator uses the latest available control input.
System Description

1. $\tau_{k}^{sc}$: delay from sensor to controller module for the $k$th network packet;
2. $\tau_{k}^{c}$: computation delay for the $k$th network packet;
3. $\tau_{k}^{ca}$: delay from controller to actuator module for the $k$th network packet;
4. $\tau_{k}$: total delay from sensor to actuator module for the $k$th network packet.
5. The switches $S_1$ and $S_2$ model the possibility of packet loss.
Sensor module samples data at instants $nh$;

2. $i_k$ denotes the $n$th sample number which is carried by the $k$th received network packet at the actuator’s input;

3. $\{i_1, \ldots, i_n, \ldots\} = \{1, \ldots, n, \ldots\} \rightarrow$ no packet dropout or disordering occurred;

4. $i_{k+1} \neq i_k + 1 \rightarrow$ a transmission failure occurred.
Plant’s model
\[ \dot{x}_p(t) = A_p x_p(t) + B_p u_p(t), \quad (1) \]

\[ y_p(t) = C_p x_p(t), \quad (2) \]

Controller’s input
\[ u_c(t) = y_p(i_k h) = C_p x_p(i_k h), \quad \forall k \in \mathbb{N}^*, \quad (3) \]

Plant’s input
\[ u_p(t) = y_c(i_k h + \tau_{sc}^k + \tau_{c}^k + \tau_{ca}^k) = K C_p x_p(i_k h), \quad (4) \]

Delay constraints

\[ (i_{k+1} - i_k)h + \tau_{k+1} \leq \eta, \quad \text{(upper bound)} \]

\[ \tau \leq \tau_k, \quad \forall k \in \mathbb{N}^*, \quad \text{(lower bound)} \]

\[ d(t) = t - i_k h, \quad \dot{d}(t) = 1. \]
System Description

Closed loop system

\[ \dot{x}(t) = Ax(t) + A_d x(t - d(t)), \quad (5) \]

\[ x(t) = \phi(t), \quad t \in [t_1 - \eta, t_1] \quad (6) \]

\[ \tau \leq d(t) \leq \eta \quad (7) \]

where

\[ A_d = B_p K C_p \]

\begin{align*}
\Delta A &= M_A F_A N_A, \quad (9) \\
\Delta A_d &= M_{Ad} F_{Ad} N_{Ad} \quad (10)
\end{align*}

with

\[ F_A^T F_A \leq I, \]

\[ F_{Ad}^T F_{Ad} \leq I. \]
Considering the Lyapunov function candidate

\[ V(t) = \sum_{i=1}^{3} V_i(t), \quad (11) \]

where

\[ V_1(t) = x^T(t) P x(t), \]

\[ V_2(t) = \int_{t-\tau}^{t} x(s)^T Q_1 x(s) \, ds + \int_{t-\eta}^{t} x(s)^T Q_2 x(s) \, ds + \int_{t-\alpha d(t)}^{t} x(s)^T Q_3 x(s) \, ds, \quad (12) \]

\[ V_3(t) = \int_{-\eta}^{0} \int_{t+\beta}^{t} \dot{x}(s)^T Z_1 \dot{x}(s) \, dsd\beta \]

\[ + \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{x}(s)^T Z_2 \dot{x}(s) \, dsd\beta, \]

and matrices \( P = P^T > 0, \ Q_i = Q_i^T \geq 0, \ Z_j = Z_j^T > 0, \ i \in \{1, 2, 3\}, \ j \in \{1, 2\}, \) the following stability criterion was derived.
Theorem (1)

For given scalars $0 \leq \tau < \eta, 0 < \alpha < 1, \beta_A > 0, \beta_{Ad} > 0$ and $\varepsilon > 0$ such that

$$
\begin{bmatrix}
M_A^T U M_A & M_A^T U M_{Ad} \\
M_{Ad}^T U M_A & M_{Ad}^T U M_{Ad}
\end{bmatrix} - \varepsilon^{-1} I < 0,
$$

the NCS described by (8) is asymptotically stable if there exist matrices $P = P^T > 0, Q_i = Q_i^T \geq 0, Z_j = Z_j^T > 0, i \in \{1, 2, 3\}, j \in \{1, 2\}$ such that

$$
\begin{bmatrix}
G^T X + XG + K & XB \\
B^T X & -I
\end{bmatrix} < 0. \quad (13)
$$

holds, where
Robust stability criterion for NCSs

\[ U = \eta Z_1 + (\eta - \tau) Z_2, \]
\[ R_q = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} A^T U & M^T \end{bmatrix} A \\ A_d^T U & M_{Ad} \end{bmatrix} \end{bmatrix}, \]
\[ Q_q = \begin{bmatrix} -\varepsilon^{-1} I + \begin{bmatrix} M^T \\ M_{Ad} \end{bmatrix} U \begin{bmatrix} M_A & M_{Ad} \end{bmatrix} \end{bmatrix}, \]
\[ A_q = \begin{bmatrix} 0 & 0 \\ \frac{1}{\eta - \tau} Z_2 & \frac{1}{\eta - \tau} (Z_1 + Z_2) \end{bmatrix}, \]
\[ Z_q = \begin{bmatrix} Z_{q11} & 0 & 0 \\ 0 & Z_{q22} & 0 \\ 0 & 0 & Z_{q33} \end{bmatrix}, \]
\[ Z_{q11} = -Q_1 - \frac{1}{\eta - \tau} Z_2, \]
\[ Z_{q22} = -Q_2 - \frac{1}{\eta - \tau} (Z_1 + Z_2), \]
\[ Z_{q33} = -(1 - \alpha) Q_3 - \frac{1}{\alpha \eta} Z_1 - \frac{1}{(1 - \alpha) \eta} Z_1, \]
\[ X = \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ G = \begin{bmatrix} A & A_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ B = \begin{bmatrix} M_A \sqrt{\beta_A} & M_{Ad} \sqrt{\beta_{Ad}^{-1}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ K = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{12}^T & K_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_q & R_q \\ A_q^T & Z_q & 0 \\ R_q^T & 0 & Q_q \end{bmatrix}, \]
\[ K_{11} = A^T U A + \varepsilon^{-1} N_A^T N_A - \frac{1}{\alpha \eta} Z_1 \]
\[ + Q_1 + Q_2 + Q_3 + \beta_A^{-1} N_A^T N_A, \]
\[ K_{12} = A^T U A_d, \]
\[ K_{22} = A_d^T U A_d + \varepsilon^{-1} N_{Ad}^T N_{Ad} - \frac{1}{(1 - \alpha) \eta} Z_1 \]
\[ - \frac{1}{\eta - \tau} Z_2 - \frac{1}{\eta - \tau} (Z_1 + Z_2) + \beta_{Ad} N_{Ad}^T N_{Ad}. \]
Numerical Examples

Problem

What is the maximum delay that we guarantee will keep the system stable?

\[
\dot{x}(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - d(t)),
\]

\[
\tau \leq d(t) \leq \eta.
\]

\[
A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix},
\]

\[
A_d = \begin{bmatrix} -1.4 & 0 \\ -0.8 & -1.5 \end{bmatrix},
\]

\[
\Delta A = 0, \quad \Delta A_d = 0.
\]

Table 1: Maximum delay’s upper bounds for different criteria (\(\tau = 0.4s\))

<table>
<thead>
<tr>
<th></th>
<th>(\eta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yue et al. (2005)</td>
<td>1.13s</td>
</tr>
<tr>
<td>He et al. (2007)</td>
<td>1.16s</td>
</tr>
<tr>
<td>Zhu et al. (2008)</td>
<td>1.17s*</td>
</tr>
<tr>
<td>Theorem (1)(^a)</td>
<td>1.17s*</td>
</tr>
</tbody>
</table>

* The result in Zhu et al. (2008) is a corollary of Theorem (1) when no uncertainties are considered.

\(^a\)Choosing \(\alpha = 0.75\).
**Problem**

What is the maximum delay that we guarantee will keep the system stable?

\[
\dot{x}(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - d(t)), \\
\tau \leq d(t) \leq \eta.
\]

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},
\]

\[
M_A = N_A = \begin{bmatrix} \sqrt{0.3} & 0 \\ 0 & \sqrt{0.2} \end{bmatrix},
\]

\[
M_{A_d} = N_{A_d} = \begin{bmatrix} \sqrt{0.2} & 0 \\ 0 & \sqrt{0.3} \end{bmatrix},
\]

\[
F_A(t) = F_{A_d}(t) = \begin{bmatrix} \cos t & 0 \\ 0 & \sin t \end{bmatrix}.
\]

**Table 2:** Maximum delay’s upper bounds for different criteria ($\tau = 0.4\text{s}$)

<table>
<thead>
<tr>
<th></th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Su et al. (1992)</td>
<td>0.1575s</td>
</tr>
<tr>
<td>Xu (1994)</td>
<td>0.1575s</td>
</tr>
<tr>
<td>Cao et al. (1998)</td>
<td>0.2558s</td>
</tr>
<tr>
<td>Jing (2004)</td>
<td>0.3916s</td>
</tr>
<tr>
<td>Yan et al. (2008)</td>
<td>0.6090s</td>
</tr>
<tr>
<td>Theorem (1)$^b$</td>
<td>0.6847s</td>
</tr>
</tbody>
</table>

$^b$ Choosing $\alpha = 0.6$, $\epsilon = 0.9$, $\beta_A = \beta_{A_d} = 0.8$. 
Problem

What is the minimum delay’s influence?

\[ \dot{x}(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - d(t)), \]
\[ \tau \leq d(t) \leq \eta. \]

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},
A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},
\]

\[
M_A = N_A = \begin{bmatrix} \sqrt{0.3} & 0 \\ 0 & \sqrt{0.2} \end{bmatrix},
\]

\[
M_{A_d} = N_{A_d} = \begin{bmatrix} \sqrt{0.2} & 0 \\ 0 & \sqrt{0.3} \end{bmatrix},
\]

\[
F_A(t) = F_{A_d}(t) = \begin{bmatrix} \cos t & 0 \\ 0 & \sin t \end{bmatrix}.
\]

Table 3: Maximum delay’s upper bounds for Theorem (1) \(^c\)

<table>
<thead>
<tr>
<th>(\tau)</th>
<th>(\eta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1s</td>
<td>0.6977s</td>
</tr>
<tr>
<td>0.2s</td>
<td>0.7133s</td>
</tr>
<tr>
<td>0.3s</td>
<td>0.7336s</td>
</tr>
<tr>
<td>0.4s</td>
<td>0.7590s</td>
</tr>
</tbody>
</table>

\(^c\) Choosing \(\alpha = 0.6, \epsilon = 0.9, \beta_A = \beta_{A_d} = 0.8.\)
The new robust stability criterion presented here is able to deal with:
- Bounded model uncertainties;
- Time-varying network delay;
- Packet losses.

The theorem presented gives less conservative results compared with previous works for the network delay’s upper bound. It can also be seen as an extension of previous criteria that assumed perfect model’s knowledge;

Results concerning the system’s stabilization will be published soon.
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Questions?