Automatic Parallelization of Divide and Conquer Algorithms

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Outline

- Example
- Information required to parallelize divide and conquer algorithms
- How compiler extracts parallelism
  - Key technique: constraint systems
- Results
- Related work
- Conclusion

Example - Divide and Conquer Sort

7	4	6	1	3	5	8	2

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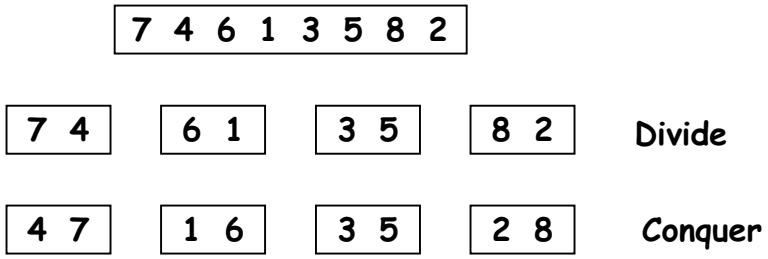
Divide

Example - Divide and Conquer Sort



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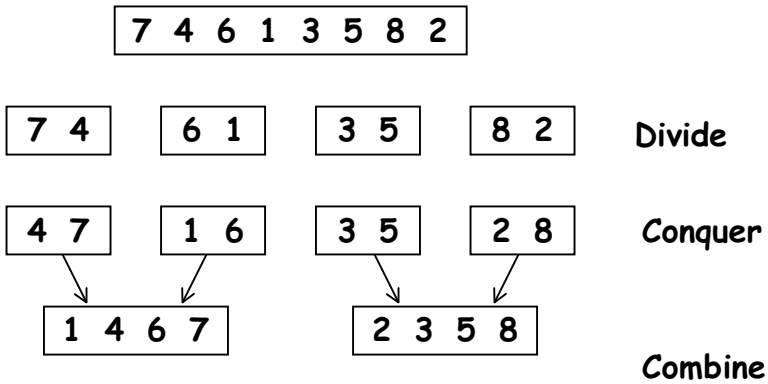
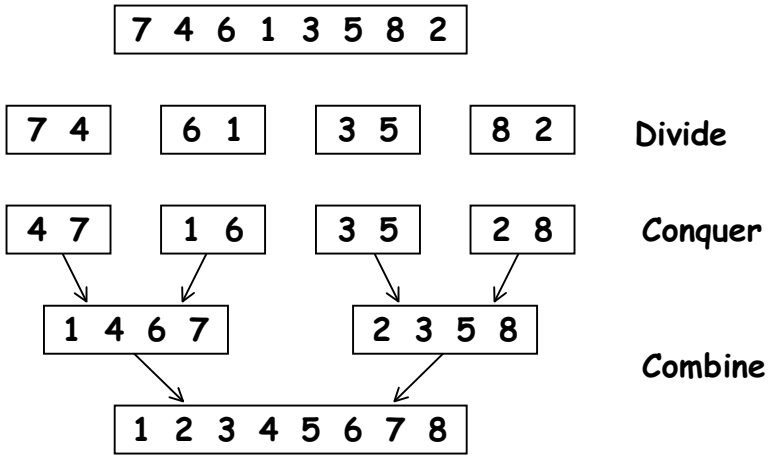


Example - Divide and Conquer Sort



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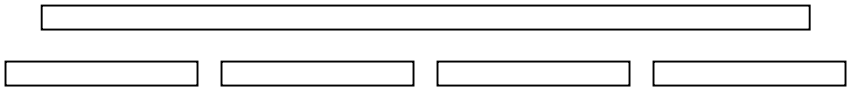

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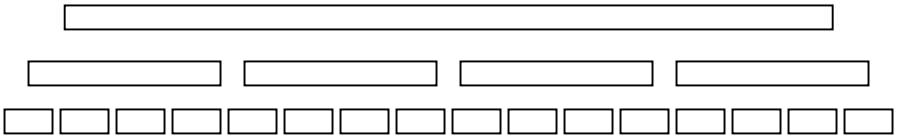

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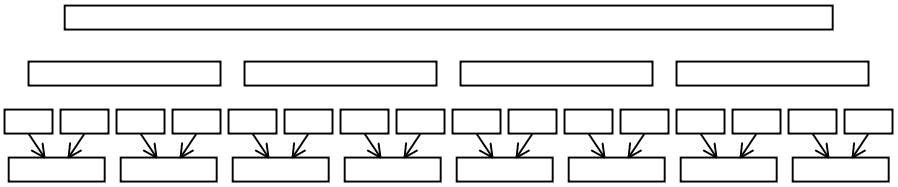

Divide and Conquer Algorithms

- Lots of Recursively Generated Concurrency
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  - Combine Results in Parallel
- Good Cache Performance
  - Problems Naturally Scale to Fit in Cache
  - No Cache Size Constants in Code

Divide and Conquer Algorithms

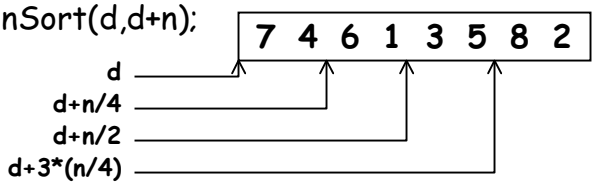
- Lots of Recursively Generated Concurrency
  - Recursively Solve Subproblems in Parallel
  - Combine Results in Parallel
- Good Cache Performance
  - Problems Naturally Scale to Fit in Cache
  - No Cache Size Constants in Code
- Lots of Programs
  - Sort Programs
  - Dense Matrix Programs

```
  "Sort n Items in d, Using t as Temporary Storage"
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
       sort(d,t,n/4);
       sort(d+n/4,t+n/4,n/4);
       sort(d+n/2,t+n/2,n/4);
       sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
       merge(d,d+n/4,d+n/2,t);
       merge(d+n/2,d+3*(n/4),d+n,t+n/2);
       merge(t,t+n/2,t+n,d);
   } else insertionSort(d,d+n);
```

"Recursively Sort Four Quarters of d"

```
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
  ⇨ sort(d,t,n/4);
  ⇨ sort(d+n/4,t+n/4,n/4);
  ⇨ sort(d+n/2,t+n/2,n/4);
  ⇨ sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
    merge(d,d+n/4,d+n/2,t);
    merge(d+n/2,d+3*(n/4),d+n,t+n/2);
    merge(t,t+n/2,t+n,d);
  } else insertionSort(d,d+n);
```

Subproblems Identified Using Pointers Into Middle of Array



"Recursively Sort Four Quarters of d"

```
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
  ⇨ sort(d,t,n/4);
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  ⇨ sort(d+n/2,t+n/2,n/4);
  ⇨ sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
    merge(d,d+n/4,d+n/2,t);
    merge(d+n/2,d+3*(n/4),d+n,t+n/2);
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  } else insertionSort(d,d+n);
```
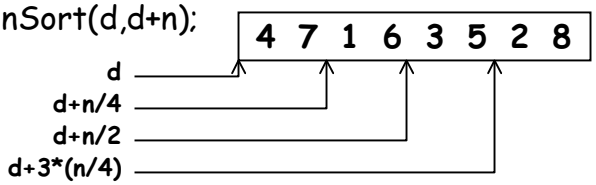

Sorted Results Written Back Into Input Array



"Merge Sorted Quarters of d Into Halves of t"

```
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
       sort(d,t,n/4);
       sort(d+n/4,t+n/4,n/4);
       sort(d+n/2,t+n/2,n/4);
       sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
    ⇨ merge(d,d+n/4,d+n/2,t);
    ⇨ merge(d+n/2,d+3*(n/4),d+n,t+n/2);
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    } else insertionSort(d,d+n);
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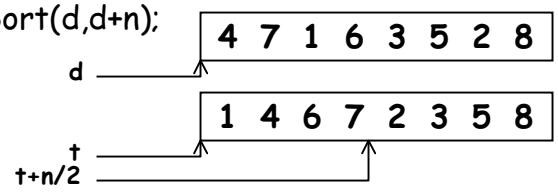

"Merge Sorted Halves of t Back Into d"

```
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
       sort(d,t,n/4);
       sort(d+n/4,t+n/4,n/4);
       sort(d+n/2,t+n/2,n/4);
       sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
       merge(d,d+n/4,d+n/2,t);
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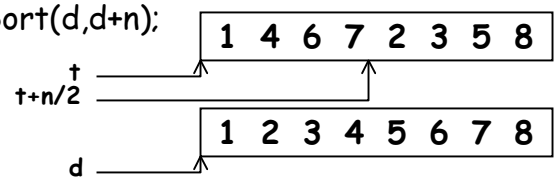

"Use a Simple Sort for Small Problem Sizes"

```
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
       sort(d,t,n/4);
       sort(d+n/4,t+n/4,n/4);
       sort(d+n/2,t+n/2,n/4);
       sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
       merge(d,d+n/4,d+n/2,t);
       merge(d+n/2,d+3*(n/4),d+n,t+n/2);
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⇨  } else insertionSort(d,d+n);
```

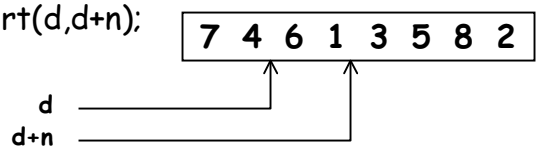

"Use a Simple Sort for Small Problem Sizes"

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       sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
       merge(d,d+n/4,d+n/2,t);
       merge(d+n/2,d+3*(n/4),d+n,t+n/2);
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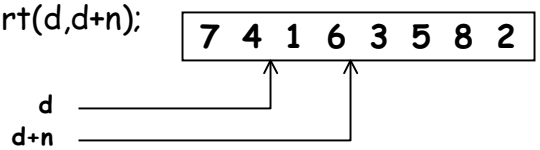

Parallel Execution

```
void sort(int *d, int *t, int n)
    if (n > CUTOFF) {
       spawn sort(d,t,n/4);
       spawn sort(d+n/4,t+n/4,n/4);
       spawn sort(d+n/2,t+n/2,n/4);
       spawn sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));
       sync;
       spawn merge(d,d+n/4,d+n/2,t);
       spawn merge(d+n/2,d+3*(n/4),d+n,t+n/2);
       sync;
       merge(t,t+n/2,t+n,d);
   } else insertionSort(d,d+n);
```

What Do You Need to Know to Exploit this Form of Parallelism?

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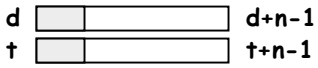
Calls to sort access disjoint parts of d and t

Together, calls access [d,d+n-1] and [t,t+n-1]

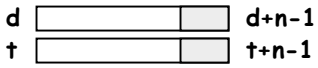
sort(d,t,n/4);



sort(d+n/4,t+n/4,n/4);

sort(d+n/2,t+n/2,n/4);

sort(d+3*(n/4),t+3*(n/4),n-3*(n/4));

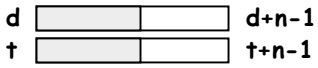


What Do You Need to Know to Exploit this Parallelism?

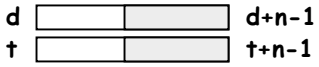
First two calls to merge access disjoint parts of d,t

Together, calls access [d,d+n-1] and [t,t+n-1]

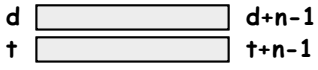
merge(d,d+n/4,d+n/2,t);



merge(d+n/2,d+3*(n/4),d+n,t+n/2);



merge(t,t+n/2,t+n,d);



What Do You Need to Know to Exploit this Parallelism?

Calls to insertionSort access [d,d+n-1]

insertionSort(d,d+n);

What Do You Need to Know to Exploit this Parallelism?

The Regions of Memory Accessed by Complete Executions of Procedures

How Hard Is it to Extract these Regions?

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Challenging

How Hard Is it to Extract these Regions?

```
insertionSort(int *l, int *h) {
  int *p, *q, k;
  for (p = l+1; p < h; p++) {
      for (k = *p, q = p-1; l <= q && k < *q; q--)
            *(q+1) = *q;
      *(q+1) = k;
  }
}
```

Not Immediately Obvious That insertionSort(l,h) Accesses [l,h-1]

How Hard Is it to Extract these Regions?

```
merge(int *l1, int*m, int *h2, int *d) {
  int *h1 = m; int *l2 = m;
  while ((l1 < h1) && (l2 < h2))
      if (*l1 < *l2) *d++ = *l1++;
      else *d++ = *l2++;
  while (l1 < h1) *d++ = *l1++;
  while (l2 < h2) *d++ = *l2++;
}
```

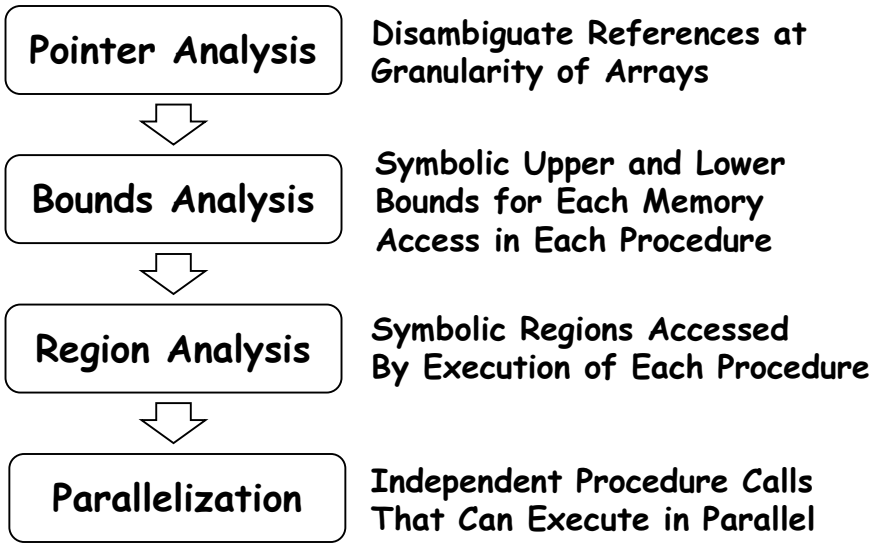
Not Immediately Obvious That merge(l,m,h,d) Accesses [l,h-1] and [d,d+(h-l)-1]

Issues

- Pervasive Use of Pointers
  - Pointers into Middle of Arrays
  - Pointer Arithmetic
  - Pointer Comparison
- Multiple Procedures
  - sort(int *d, int *t, n)
  - insertionSort(int *l, int *h)
  - merge(int *l, int *m, int *h, int *t)
- Recursion

How The Compiler Does It

Structure of Compiler



Example

```
f(char *p, int n)
  if (n > CUTOFF) {
      f(p, n/2);          initialize first half
      f(p+n/2, n/2);      initialize second half
  } else {
      base case: initialize small array
      int i = 0;
      while (i < n) { *(p+i) = 0; i++; }
  }
```

Bounds Analysis

- For each variable at each program point, derive upper and lower bounds for value
- Bounds are symbolic expressions
  - symbolic variables in expressions represent initial values of parameters
  - linear combinations of these variables
  - multivariate polynomials

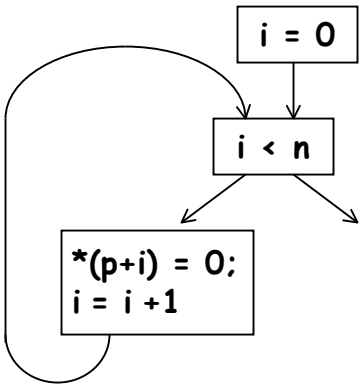
Bounds Analysis

What are upper and lower bounds for region accessed by while loop in base case?

```
int i = 0;
while (i < n) { *(p+i) = 0; i++; }
```

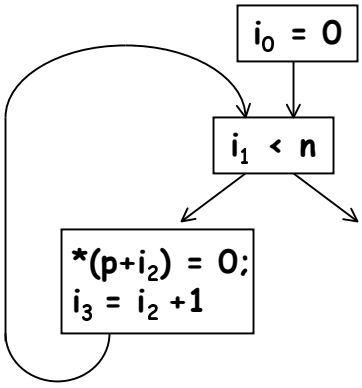
Bounds Analysis, Step 1

Build control flow graph



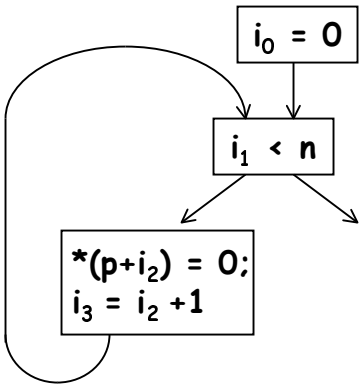
Bounds Analysis, Step 2

Number different versions of variables



Bounds Analysis, Step 3

Set up constraints for lower bounds



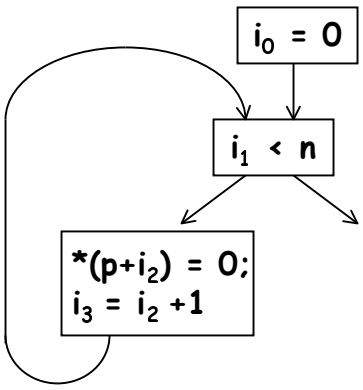
$l(i_0) <= 0$

$l(i_1) <= l(i_0)$
$l(i_1) <= l(i_3)$

$l(i_2) <= l(i_1)$
$l(i_3) <= l(i_2)+1$

Bounds Analysis, Step 3

Set up constraints for lower bounds



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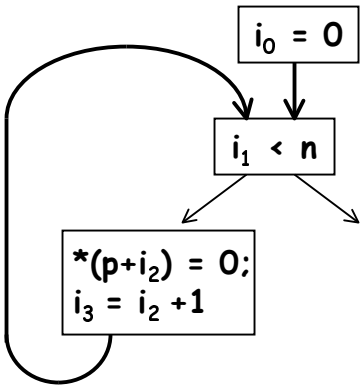


$l(i_0) <= 0$

$l(i_1) <= l(i_0)$ ⇦

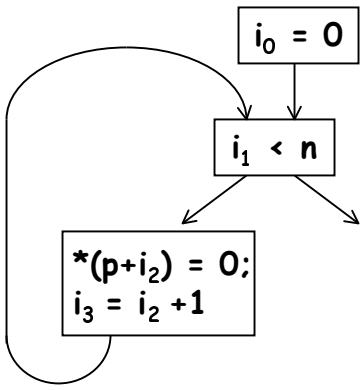
$l(i_1) <= l(i_3)$ ⇦

$l(i_2) <= l(i_1)$

$l(i_3) <= l(i_2)+1$

Bounds Analysis, Step 4

Set up constraints for upper bounds



$l(i_0) <= 0$

$l(i_1) <= l(i_0)$

$l(i_1) <= l(i_3)$

$l(i_2) <= l(i_1)$

$l(i_3) <= l(i_2)+1$

$0 <= u(i_0)$

$u(i_0) <= u(i_1)$

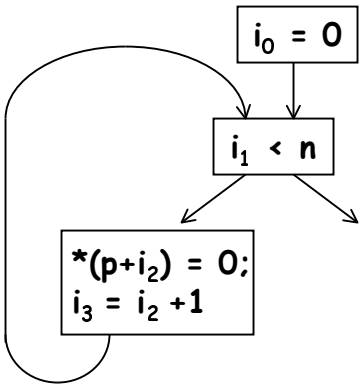
$u(i_3) <= u(i_1)$

$\min(u(i_1),n-1) <= u(i_2)$

$u(i_2)+1 <= u(i_3)$

Bounds Analysis, Step 4

Set up constraints for upper bounds



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$l(i_1) <= l(i_0)$
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$l(i_2) <= l(i_1)$
$l(i_3) <= l(i_2)+1$

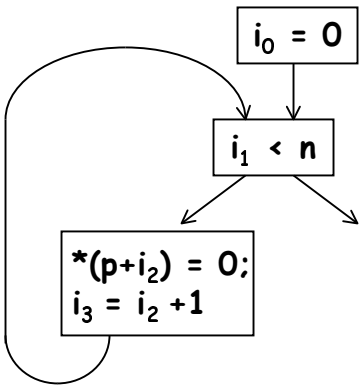
$0 <= u(i_0)$

$u(i_0) <= u(i_1)$
$u(i_3) <= u(i_1)$

$\min(u(i_1),n-1) <= u(i_2)$ ⇦
$u(i_2)+1 <= u(i_3)$

Bounds Analysis, Step 4

Set up constraints for upper bounds



$l(i_0) <= 0$

$l(i_1) <= l(i_0)$
$l(i_1) <= l(i_3)$

$l(i_2) <= l(i_1)$
$l(i_3) <= l(i_2)+1$

$0 <= u(i_0)$

$u(i_0) <= u(i_1)$
$u(i_3) <= u(i_1)$

$n-1 <= u(i_2)$ ⇦
$u(i_2)+1 <= u(i_3)$

Bounds Analysis, Step 5

Generate symbolic expressions for bounds

Goal: express bounds in terms of parameters

$$l(i_0) = c_1p + c_2n + c_3$$
$$l(i_1) = c_4p + c_5n + c_6$$
$$l(i_2) = c_7p + c_8n + c_9$$
$$l(i_3) = c_{10}p + c_{11}n + c_{12}$$

$$u(i_0) = c_{13}p + c_{14}n + c_{15}$$
$$u(i_1) = c_{16}p + c_{17}n + c_{18}$$
$$u(i_2) = c_{19}p + c_{20}n + c_{21}$$
$$u(i_3) = c_{22}p + c_{23}n + c_{24}$$

Bounds Analysis, Step 6

Substitute expressions into constraints

$$c_1p + c_2n + c_3 \le 0$$

$$c_4p + c_5n + c_6 \le c_1p + c_2n + c_3$$
$$c_4p + c_5n + c_6 \le c_{10}p + c_{11}n + c_{12}$$

$$c_7p + c_8n + c_9 \le c_4p + c_5n + c_6$$
$$c_{10}p + c_{11}n + c_{12} \le c_7p + c_8n + c_9 + 1$$

$$0 \le c_{13}p + c_{14}n + c_{15}$$

$$c_{13}p + c_{14}n + c_{15} \le c_{16}p + c_{17}n + c_{18}$$
$$c_{22}p + c_{23}n + c_{24} \le c_{16}p + c_{17}n + c_{18}$$

$$n - 1 \le c_{19}p + c_{20}n + c_{21}$$
$$c_{19}p + c_{20}n + c_{21} + 1 \le c_{22}p + c_{23}n + c_{24}$$

Goal

Solve Symbolic Constraint System

find values for constraint variables c_1, ..., c_{24}
that satisfy the inequality constraints

Maximize Lower Bounds

Minimize Upper Bounds

Bounds Analysis, Step 7

Apply expression ordering principle

$c_1p + c_2n + c_3 <= c_4p + c_5n + c_6$

If

$c_1 <= c_4$, $c_2 <= c_5$, and $c_3 <= c_6$

Bounds Analysis, Step 7

Apply expression ordering principle

Generate a linear program

$c_1 <= 0$ $c_2 <= 0$ $c_3 <= 0$ $0 <= c_{13}$ $0 <= c_{14}$ $0 <= c_{15}$

$c_4 <= c_1$ $c_5 <= c_2$ $c_6 <= c_3$ $c_{13} <= c_{16}$ $c_{14} <= c_{17}$ $c_{15} <= c_{18}$

$c_4 <= c_{10}$ $c_5 <= c_{11}$ $c_6 <= c_{12}$ $c_{22} <= c_{16}$ $c_{23} <= c_{17}$ $c_{24} <= c_{18}$

$c_7 <= c_4$ $c_8 <= c_5$ $c_9 <= c_6$ $0 <= c_{19}$ $1 <= c_{20}$ $-1 <= c_{21}$

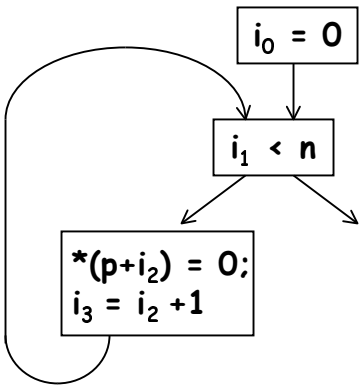
$c_{10} <= c_7$ $c_{11} <= c_8$ $c_{12} <= c_9+1$ $c_{19} <= c_{22}$ $c_{20} <= c_{23}$ $c_{21}+1 <= c_{24}$

Objective Function:

$$\max \underbrace{(c1 + \cdots + c12)}_{\text{lower bounds}} - \underbrace{(c13 + \cdots + c24)}_{\text{upper bounds}}$$

Bounds Analysis, Step 8

Solve linear program to extract bounds



$l(i_0) = 0$
$l(i_1) = 0$
$l(i_2) = 0$
$l(i_3) = 0$

$u(i_0) = 0$
$u(i_1) = n$
$u(i_2) = n-1$
$u(i_3) = n$

Region Analysis

Goal: Compute Accessed Regions of Memory

- Intra-Procedural
  - Use bounds at each load or store
  - Compute accessed region
- Inter-Procedural
  - Use intra-procedural results
  - Set up another constraint system
  - Solve to find regions accessed by entire execution of the procedure

Basic Principle of Inter-Procedural Region Analysis

- For each procedure
  - Generate symbolic expressions for upper and lower bounds of accessed regions
- Constraint System
  - Accessed regions include regions accessed by statements in procedure
  - Accessed regions include regions accessed by invoked procedures

Inter-Procedural Constraints in Example

```
f(char *p, int n)
  if (n > CUTOFF) {
      f(p, n/2);                    l(f,p,n) <= l(f,p,n/2)
                                    u(f,p,n) <= u(f,p,n/2)
      f(p+n/2, n/2);                l(f,p,n) <= l(f,p+n/2,n/2)
                                    u(f,p,n) <= u(f,p+n/2,n/2)
  } else {
      int i = 0;
      while (i < n) {
         *(p+i) = 0; i++;           l(f,p,n) <= p
      }                             u(f,p,n) <= p+n-1
  }
```

Derive Constraint System

- Generate symbolic expressions
  - $l(f,p,n) = C_1p + C_2n + C_3$
  - $u(f,p,n) = C_4p + C_5n + C_6$
- Build constraint system
  - $C_1p + C_2n + C_3 <= p$
  - $C_4p + C_5n + C_6 <= p + n - 1$
  - $C_1p + C_2n + C_3 <= C_1p + C_2(n/2) + C_3$
  - $C_4p + C_5n + C_6 <= C_4p + C_5(n/2) + C_6$
  - $C_1p + C_2n + C_3 <= C_1(p+n/2) + C_2(n/2) + C_3$
  - $C_4p + C_5n + C_6 <= C_4(p+n/2) + C_5(n/2) + C_6$

Solve Constraint System

- Simplify Constraint System
  - $C_1p + C_2n + C_3 <= p$
  - $C_4p + C_5n + C_6 <= p + n - 1$
  - $C_2n <= C_2(n/2)$
  - $C_5n <= C_5(n/2)$
  - $C_2(n/2) <= C_1(n/2)$
  - $C_5(n/2) <= C_4(n/2)$
- Generate and Solve Linear Program
  - l(f,p,n) = p
  - u(f,p,n) = p+n-1

Parallelization

- Dependence Testing of Two Calls
  - Do accessed regions intersect?
  - Based on comparing upper and lower bounds of accessed regions
  - Comparison done using expression ordering principle
- Parallelization
  - Find sequences of independent calls
  - Execute independent calls in parallel

Details

- Inter-procedural positivity analysis
  - Verify that variables are positive
  - Required for correctness of expression ordering principle
- Correlation Analysis
- Integer Division
  - Basic Idea : $(n-1)/2 <= \lfloor n/2 \rfloor <= n/2$
  - Generalized : $(n-m+1)/m <= \lfloor n/m \rfloor <= n/m$
- Linear System Decomposition

Experimental Results

- Implementation - SUIF, lp_solve, Cilk

Speedup for Matrix Multiply

Speedup for Sort

Thanks: Darko Marinov, Nate Kushman, Don Dailey

Related Work

- Shape Analysis
  - Chase, Wegman, Zadek (PLDI 90)
  - Ghiya, Hendren (POPL 96)
  - Sagiv, Reps, Wilhelm (TOPLAS 98)
- Commutativity Analysis
  - Rinard and Diniz (PLDI 96)
- Predicated Dataflow Analysis
  - Moon, Hall, Murphy (ICS 98)

Related Work

- Array Region Analysis
  - Triolet, Irigoin and Feautrier (PLDI 86)
  - Havlak and Kennedy (IEEE TPDS 91)
  - Hall, Amarasinghe, Murphy, Liao and Lam (SC 95)
  - Gu, Li and Lee (PPoPP 97)
- Symbolic Analysis of Loop Variables
  - Blume and Eigenmann (IPPS 95)
  - Haghigat and Polychronopoulos (LCPC 93)

Future

- Static Race Detection for Explicitly Parallel Programs
- Static Elimination of Array Bounds Checks
- Static Pointer Validation Checks
- Result:
  - Safety Guarantees
  - No Efficiency Compromises

Context

- Mainstream Parallelizing Compilers
  - Loop Nests, Dense Matrices
  - Affine Access Functions
  - Key Problem:Solving Diophantine Equations
- Compilers for Divide and Conquer Algorithms
  - Recursion, Dense Arrays (dynamic)
  - Pointers, Pointer Arithmetic
  - Key Problems: Pointer Analysis, Symbolic Region Analysis, Solving Linear Programs