Tyepartate Checking and Regular Graph Constraints
Viktor Kuncak and Martin Rinard
Laboratory for Computer Science
Massachusetts Institute of Technology
Cambridge, MA 02139
{vkuncak,rinard}@lcs.mit.edu
MIT-LCS-TR-863, September 2002

Abstract
We introduce regular graph constraints and explore their decidability properties. The motivation for regular graph constraints is 1) type checking of changing types of objects in the presence of linked data structures, 2) shape analysis techniques, and 3) generalization of similar constraints over trees and grids.

Tyepartate checking for recursive and potentially cyclic data structures requires verifying the validity of implication for regular graph constraints. The implication of regular graph constraints also arises in shape analysis algorithms such as role-analysis and some analyses based on three-valued logic.

Over the class of lists regular graph constraints reduce to a nondeterministic finite state automaton as a special case. Over the class of trees the constraints reduce to a nondeterministic top-down tree automaton, and over the class of grids our constraints reduce to domino system and tiling problems.

We define a subclass of graphs called heaps as an abstraction of the data structures that a program constructs during its execution. We show that satisfiability of regular graph constraints over the class of heaps is decidable. However, determining the validity of implication for regular graph constraints over the class of heaps is undecidable. The undecidability of implication is the central result of the paper. The result is somewhat surprising because our simple constraints are strictly less expressive than existential monadic second-order logic over graphs. In the key step of our proof we introduce the class of corresponder graphs which mimic solutions of Post correspondence problem instances. We show undecidability by exhibiting a characterization of corresponder graphs in terms of presence and absence of homomorphisms to a finite number of fixed graphs.

The undecidability of implication of regular graph constraints implies that there is no algorithm that will verify that procedure preconditions are met or that the invariants are maintained when these properties are expressed in any specification language at least as expressive as regular graph constraints.

*This research was supported in part by DARPA Contract F30602-00-C-1992, NSF Grant CCR00-36154, NSF Grant CCR00-63513, and the Singapore-MIT Alliance.

Keywords: Type Checking, Shape Analysis, Program Verification, Graph Homomorphism, Post Correspondence Problem, Monadic Second-Order Logic

1 Introduction
Types capture important properties of objects in the program. In an imperative language properties of objects change over time. It is therefore desirable that types capture changing properties of objects. A tyepartate system is a system where types of objects change over time. A simple tyepartate system was introduced in [25], more recent examples include [14, 23, 28, 5]. We view tyepartate as a step towards statically checking properties of objects [17, 8].

One of the difficulties with defining object properties in object-oriented languages is that a property of an object may depend on properties of other objects in the heap. Some systems allow programmers to identify properties of an object x in terms of properties of objects y such that x references y. The idea that important properties of an object x may depend on properties of objects z such that z references x was introduced in the role system [14].

In general, properties of objects may be mutually recursive and the referencing graph of objects may be cyclic. Due to cycles in the heap the least fixpoint solution for the recursive object properties is not acceptable because there is no basis to ground the inductive definitions of these properties. We therefore say that a heap satisfies a set of properties if there exists some choice of predicates that satisfy the mutually recursive definitions. The existential quantification over predicates leads to constraints that have the form of existential monadic second-order logic [7]. For a presentation of role analysis and related systems from the perspective of monadic second-order logic, see [15].

In this paper we present a very simple form of constraints that we call regular graph constraints. A set of regular graph constraints can be specified by a single graph G. A heap H satisfies the constraints if there exists a graph homomorphism from H to G. Regular graph constraints abstract the problem of mutually recursive definitions of properties over potentially cyclic graphs. The existential quantification over predicates is modeled in regular graph constraints as the existence of a homomorphism to a given fixed graph. Regular graph constraints are closed under conjunction and in certain cases are closed under disjunction (Section 2.9). Moreover, regular graph constraints generalize the notion of tree
automata [27, 3] and domino systems [12], without going all the way to monadic second-order logic for richer domains. In this paper we consider as the domain of interpretation the class of heaps. Our notion of heap is an abstraction of garbage collected heap in a programming languages like Java or ML. Heaps contain a "root" node and a "null" node, all nodes are reachable from the root, and all edges are total functions mapping nodes to nodes.

In Section 2.8 we show that there is a simple and efficient algorithm that decides if regular graph constraints have a heap model. This results in a simple sanity test on regular graph constraint specifications that rules out the contradictory specifications.

We next turn to the problem of checking if one set of regular graph constraints implies another set of regular graph constraints over the set of heaps. Our main contribution (Section 3) is the proof that the implication problem is undecidable.

The implication problem of graphs arises in compositional checking of programs if procedure preconditions or postconditions are given as regular graph constraints. In Section 3.5.4 we show that the implication problem also arises when proving that an invariant holds after every program step. These verification problems are therefore undecidable. Our result places limitations on the completeness of systems such as role analysis [14] and shape analysis [20] that use homomorphic images to represent the abstraction of the heap. The undecidability of regular graph constraints means that semantically checking the implication of such homomorphic graph images is undecidable.

A common way of showing the undecidability of problems over graphs is to encode the Turing machine computation histories [22] as a special form of graphs called grids. The difficulty with showing the undecidability of implication of regular graph constraints is that regular graph constraints cannot define the subclass of grids among the class of heaps. Indeed, this is why satisfiability of regular graph constraints over heaps is decidable. To show the undecidability of the implication of regular graph constraints, we use the constraints on both sides of the implication to restrict the set of possible counterexample models for the implication. For this purpose we introduce a new class of graphs called corresponder graphs. Satisfiability of regular graph constraints over corresponder graphs mimics the solution of a Post correspondence problem instance, and is therefore undecidable. We give a method for constructing an implication such that all counterexamples for the validity of implication are corresponder graphs which satisfy a given regular graph constraint. This shows that the validity of the implication is undecidable.

Due to closure under conjunction, the implication is reducible to the equivalence of regular graph constraints. As a result, the equivalence of two regular graph constraints is also undecidable.

2 Regular Graph Constraints

In this section we define the class of graphs considered in this paper as well as its subclasses heaps, trees, lists, grids, and corresponder graphs. We present our regular graph constraints, give several equivalent formulations of the constraints and show that our constraints capture tree automata and domino systems as special cases. We then review some decidability properties, show that satisfiability of regular constraints over heaps is efficiently decidable and state some closure properties of regular graph constraints.

2.1 Preliminaries

If \( r \subseteq A \times B \) and \( S \subseteq A \), relational image of set \( S \) under \( r \) is defined as

\[
r[S] = \{ y \mid x \in S, (x, y) \in r \}
\]

We use \( \bullet \) to mark the end of a proof and \( \oplus \) to mark the end of an example.

2.2 Graphs

We will be considering the following class of directed graphs in this paper. Our graphs contain two kinds of edges, which we represent by relations \( s_1 \) and \( s_2 \). These relations may represent fields in an object-oriented program. The constant \( \text{root} \) represents the root of the graph. We use edges terminating at null to represent partial functions and abstractions of graphs with partial functions.

**Definition 1** A graph is a relational structure

\[
G = (V, s_1, s_2, \text{null}, \text{root})
\]

where

* \( V \) is a finite set of nodes;
* \( \text{root}, \text{null} \in V \) are distinct constants, \( \text{root} \neq \text{null} \);
* \( s_1, s_2 \subseteq V \times V \) are two kinds of graph edges, such that for all nodes \( x \)

\[
\langle \text{null}, x \rangle \in s_1 \iff x = \text{null}
\]

for \( i \in 1, 2 \).

We use \( G \) to denote the class of all graphs.

An \( s_1 \)-successor of a node \( x \) is any element of the set \( s_1[\{x\}] \), similarly an \( s_2 \)-successor of \( x \) is any element of \( s_2[\{x\}] \). Note that there are exactly two edges originating from null. When drawing graphs we never show these two edges.

**Definition 2** A heap is a graph \( G = (V, s_1, s_2, \text{null}, \text{root}) \) where relations \( s_1 \) and \( s_2 \) are total functions and where for all \( x \neq \text{null} \), node \( x \) is reachable from \( \text{root} \). We use \( \mathcal{H} \) to denote the class of all heaps.

**Definition 3** The in-degree of a node \( x \) in a graph is the number of edges terminating at \( x \).

\[
\text{inDegree}(x) = \{ y \mid \exists i \ (y, x) \in s_i \}
\]

**Definition 4** A tree is a connected acyclic graph such that \( \text{inDegree}(x) \leq 1 \) for every node \( x \).

**Definition 5** A list is a tree with at most one non-null outgoing edge: for every node \( x \), \( s_1(x) = \text{null} \) or \( s_2(x) = \text{null} \).
2.3 Graphs as Constraints

A regular constraint on a graph $G$ is a constraint stating that $G$ can be homomorphically mapped to another graph $G'$.

**Definition 6** We say that a graph $G$ satisfies the constraints given by a graph $G'$, and write $G \rightarrow G'$, iff there exists a homomorphism from $G$ to $G'$.

A homomorphism between graphs is defined as follows.

**Definition 7** A function $h : V \rightarrow V'$ is a homomorphism between graphs

$$G = (V, s_1, s_2, \text{null}, \text{root})$$

and

$$G' = (V', s'_1, s'_2, \text{null}', \text{root}')$$

iff all of the following conditions hold:

1. $(x, y) \in s_i$ implies $(h(x), h(y)) \in s_i'$, for all $i \in \{1, 2\}$
2. $h(x) = \text{root}'$ iff $x = \text{root}$
3. $h(x) = \text{null}'$ iff $x = \text{null}$

If there exists a homomorphism from $G$ to $G'$, we call $G$ a model for $G'$.

We can think of a homomorphism $h : V \rightarrow V'$ as a coloring of the graph $G$. The color $h(x)$ of a node $x$ restricts the colors of the $s_1$-successors of $x$ to the colors in $s_1[\{h(x)\}]$ and the colors of the $s_2$-successors to the colors in $s_2[\{h(x)\}]$.

**Example 8** A graph $G$ can be colored by $k$ colors so that the adjacent nodes have different colors iff $G$ is homomorphic to a complete graph without self-loops,$G' = (V', s'_1, s'_2, \text{null}', \text{root}')$

with $V' = \{1, \ldots, k\}$, and

$$s'_1 = s'_2 = \{(x', y') \mid x' \neq y'\}.$$

The identity function is a homomorphism from the graph to itself. Therefore, $G \rightarrow G$ for every graph $G$. The following fundamental property of homomorphisms also holds.

**Proposition 9 (Homomorphisms compose)** Let

$$G = (V, s_1, s_2, \text{null}, \text{root})$$

$$G' = (V', s'_1, s'_2, \text{null}', \text{root}')$$

and let $h : G \rightarrow G'$ and $h' : G' \rightarrow G''$ be homomorphisms. Then $h \circ h' : G \rightarrow G''$ where $h \circ h' = h' \circ h$ is also a homomorphism.

A consequence of Proposition 9 is that $\rightarrow$ is a transitive relation.

**Definition 10 (Satisfiability)** A graph $G'$ is satisfiable over the class of graphs $C$ iff there exists a graph $G \in C$ such that $G \rightarrow G'$. Satisfiability problem over the class of graphs $C$ is: given a graph $G'$, determine if $G'$ is satisfiable.

**Definition 11 (Implication)** We say that $G_1$ implies $G_2$ over the class of graphs $C$, and write

$$G_1 \sim_C G_2,$$

iff

$$(H \rightarrow G_1) \implies (H \rightarrow G_2)$$

for all graphs $H \in C$.

We will omit $C$ in $\sim_C$ if the class of graphs is clear from the context.

The following fact provides a sufficient condition for the graph implication to hold. It is a direct consequence of Proposition 9.

**Proposition 12** Let $C$ be any class of graphs. Let $G \rightarrow G'$. Then $G \sim_C G'$.

In Section 3 we show that the implication of graphs is undecidable over the class of heaps.

### 2.4 Paths

We next state several simple properties of paths that will be useful in Section 3.

**Definition 13 (Path)** Let

$$G = (V, s_1, s_2, \text{null}, \text{root})$$

be a graph and $n \geq 0$. A path in graph $G$, denoted $p \in \text{Paths}(G)$ starting at $x_0$ and terminating at $x_n$ is a sequence of alternating nodes and labels:

$$p = x_0, l_0, x_1, l_1, \ldots, l_{n-1}, x_n$$

such that $x_0, \ldots, x_n \in V$; $l_0, \ldots, l_{n-1} \in \{1, 2\}$ and $(x_i, x_{i+1}) \in s_i$ for all $i$, $0 \leq i < n$. We define $\text{word}(p) \in \{1, 2\}^*$ by

$$\text{word}(p) = l_0 l_1 \ldots l_{n-1}.$$

**Definition 14 (Slice)** A path is a slice if it starts at root and terminates at null.

**Definition 15 (Path Image)** Let $h$ be a homomorphism from graph $G_0$ to graph $G$ and let

$$p = x_0, l_0, x_1, l_1, \ldots, l_{n-1}, x_n$$

be a path in $G_0$. Then

$$h[p] = h(x_0), l_0, h(x_1), l_1, \ldots, l_{n-1}, h(x_n)$$

is the image of path $p$ under the homomorphism $h$.

The following facts are a consequence of the definition of homomorphism.

**Proposition 16** Let $h$ be a homomorphism from graph $G_0$ to graph $G$ and let $p$ be a path in $G_0$. Then

1. $h[p]$ is a path in $G$;
2. if $p$ is a slice then $h[p]$ is a slice;
3. $\text{word}(p) = \text{word}(h[p])$. 

3
Definition 17 Let \( p \) be a path in \( G \) and \( e \) be a regular expression over the alphabet \( \{1, 2\} \). We write
\[
p \in_p e
\]
iff \( \text{word}(p) \) belongs to the language of the regular expression \( e \).

From Proposition 16 we directly obtain the following fact.

Proposition 18 (Regular Expression Test) Let \( G_0 \to G \). Then if \( e \) is any regular expression over the alphabet \( \{1, 2\} \) such that \( G_0 \) contains some slice \( p \in_p e \), then \( G \) contains some slice \( p' \in_p e \).

Proof. Let \( p \) be a slice in \( G_0 \) and \( \text{word}(p) \in_p e \). By Proposition 16 we have that \( h[p] \) is a slice in \( G \) and \( \text{word}(h[p]) = \text{word}(p) \in_p e \).

We will use the contrapositive of Proposition 18 in the proof of Proposition 37 (Section 3).

2.5 Regular Constraints and EMSOL

We can express the property of being homomorphic to a fixed graph \( G' \) by an existential monadic second-order logic formula (EMSO) of a special form.

Let
\[
G' = (V', s'_1, s'_2, \text{null}', \text{root}')
\]
be a fixed graph and let \( V' = \{x_0', \ldots, x_{k-1}'\} \) with \( x'_0 = \text{null}' \), \( x'_1 = \text{root}' \). Then (1) is a formula in EMSOL interpreted over the graph
\[
G = (V, s_1, s_2, \text{null}, \text{root})
\]
expressing that \( G \) is homomorphic to \( G' \). We use the uppercase identifiers \( X_0, \ldots, X_{k-1} \) to denote the second order variables. These variables range over the subsets of \( V \). The lowercase identifiers are first-order variables ranging over the elements of \( V \). Notation \( X_i(z) \) means that \( z \) is an element of the set \( X_i \). The predicate \( s_i(x, y) \) means that \( (x, y) \in s_i \) holds in the graph \( G \). (For the precise definition of monadic second-order logic see e.g. [10], pp28.)

\[
\exists X_0, \ldots, X_{k-1}.
\]

\[
\text{partit}(X_0, \ldots, X_{k-1}) \land \text{singl}(X_0, \text{null}) \land \text{singl}(X_1, \text{root}) \land
\forall x \forall y. \left( X_i(x) \implies P_j(x) \right)
\]

(1)

where

\[
P_j(x) = P_j^1(x) \land P_j^2(x)
\]

\[
P_j^1(x) = \forall y. \ s_i(x, y) \implies \bigvee_{0 \leq j < h} X_i(y)
\]

\[
\text{singl}(X, y) = (\forall z. X(z) \iff y = z)
\]

\[
\text{partit}(Y_0, \ldots, Y_{k-1}) =
\forall x \forall y. Y_i(x) \land \bigvee_{0 \leq i \leq n} \neg(Y_i(x) \land Y_j(x))
\]

Viewing graph as a formula justifies our previous definitions of graph satisfiability and implication. We can similarly talk about the graph conjunction, disjunction etc.

We may increase the ease of expression of some properties by relaxing the form of EMSOL formula (1) without changing the expressive power. The reason is that that the relaxed form can be converted into a normal form that can be described by a graph homomorphism. Let

\[
B_j(B_0, B_1, \ldots, B_{k-1}; A_0, \ldots, A_{m-1})
\]

denote an arbitrary propositional combination of formulas \( B_0, \ldots, B_{k-1}, A_0, \ldots, A_{m-1} \) in which \( B_0, \ldots, B_{k-1} \) occur only negatively. Then every formula of the following form is expressible as a graph constraint.

\[
\exists X_0, \ldots, X_{k-1}. \text{singl}(X_0, \text{null}) \land \text{singl}(X_1, \text{root}) \land
\forall x \forall y. \left( B_1(s_1(x, y); X_0(x), \ldots, X_{k-1}(x), X_0(y), \ldots, X_{k-1}(y)) \land
\right.
\]

\[
B_2(s_2(x, y); X_0(x), \ldots, X_{k-1}(x), X_0(y), \ldots, X_{k-1}(y))
\]

(2)

Compared to (1), the form (2) does not require \( X_0, \ldots, X_{k-1} \) to form a partition of the set of all nodes, it has the quantifiers from \( P_j \) lifted to the topmost level, and allows arbitrary propositional combinations of predicates \( X_i(x) \) and \( X_i(y) \).

Example 19 The following formula is of the form (2). Let us assume that the formula is interpreted over the class of heaps. The formula states that the node \text{root} has in-degree 0.

\[
\exists X_0, X_1, X_2. \ \text{singl}(X_0, \text{null}) \land \text{singl}(X_1, \text{root}) \land
\forall x \forall y. \left( \neg(X_1(x) \land X_2(x)) \land (X_0(x) \implies X_2(x)) \land
\right.
\]

\[
s_1(x, y) \implies \left( (X_1(x) \implies X_2(y)) \land
\right.
\]

\[
s_2(x, y) \implies \neg X_1(y)
\]

The formula uses the set \( X_2 \) that contains null as well as the nodes reachable from \text{null} along the \text{root} edges. The formula states explicitly that \( X_1 \) and \( X_2 \) are disjoint. Because \( s_1 \) edges from \( X_2 \) can only lead to \( X_3 \), there are no \( s_1 \) edges to \text{root}. The constraint that \text{root} has no \text{root} edges is specified directly, without introducing an auxiliary set of nodes. In general, negation and the implicit absence of constraints are often more convenient to express with a formula than with a graph homomorphisms.

Note that if we replaced the subformula \( (X_0(x) \implies X_2(x)) \) with \( \neg(X_0(x) \land X_2(x)) \) the resulting formula would require the existence of a cycle in the graph.

\[
\text{Proposition 20 The two families of formulas (1) and (2) denote the same family of sets of graphs.}
\]

Proof. (Sketch) Given a formula of form (1) we construct a formula of form (2) by transforming

\[
X_j(x) \implies P_j^1(x) \land P_j^2(x)
\]

into \( X_j(x) \implies P_j^1(x) \) and \( X_j(x) \implies P_j^2(x) \). This allows us to write constraints on \( s_1 \) and \( s_2 \) separately. We then lift
the universal quantification over $y$ to the top level. The partition constraint is expressible as a formula that has no occurrences of $s_i$.

Conversely, suppose we are given a formula in form (2) and suppose that the formula holds for a graph $G$ with the set of nodes $V$. This means that there exist sets $S_1, \ldots, S_{k-1}$ that satisfy $B_1$ and $B_2$. We construct a family of sets $T_0, \ldots, T_{k-1}$ that forms a partition of set $V$ such that every set $S_j$ is expressible as a union of some sets $T_i$. Namely, we define

$$T_{i_0, \ldots, i_{k-1}}(y) = S_0^{i_0} \land \cdots \land S_{k-1}^{i_{k-1}}$$

where $i_0, \ldots, i_{k-1} \in \{0, 1\}$ and

$$S^0 = S$$
$$S^1 = V \setminus S$$

This motivates a construction where the second-order variables $X_0, X_1, X_2, \ldots, X_{k-1}$ are replaced with up to $2^{2^k-2}$ new variables $X_0, X_1, Y_0, \ldots, Y_{k-1}$. We then express variables $X_0, \ldots, X_{k-1}$ in terms of $X_0, X_1, Y_0, \ldots, Y_{k-1}$, write the boolean combinations $B_i$ in disjunctive normal form and use the fact that $X_0, X_1, Y_0, \ldots, Y_{k-1}$ denote disjoint sets. As a result, it is possible to write the original formula in form (1). ■

Note that, over any class containing all heaps, not every EMSOL formula corresponds to a regular graph constraint. This is in contrast with trees [27] and grids [12]. Even first-order logic can express a constraint that the graph is a grid. On the other hand, we have shown in ([13], pp.93) that not every constraint stronger than regular graph constraints can express the gridness property.

For a normal form construction using full existential monadic second-order logic see see [21]. For using higher order logic to express heap properties more general than regular graph constraints, see [15].

2.6 Related Systems

In this section we show the relationship of our regular graph constraints with some other systems for defining sets of graphs. We also illustrate that decidability of satisfiability and implication are sensitive to the subclass of the graphs considered, and change in a non-monotonic way.

2.6.1 Words

Regular graph constraints over lists correspond to regular word languages. A regular graph constraint corresponds to a nondeterministic finite state automaton with the initial state root and the final state null.

2.6.2 Trees

Satisfiability and implication of regular graph constraints are decidable over the class of trees. The reason is that the entire MSOL is decidable over trees [27], and regular graph constraints are expressible in MSOL.

2.6.3 Pictures

Domino systems [12] are regular graph constraints over the grids.

Definition 21 A grid $m \times n$ is a graph isomorphic to

$$G = (V, s_1, s_2, \text{null, root})$$

where

$$V = \{1, \ldots, m\} \times \{1, \ldots, n\}$$

$$s_1 = \{(i, j), (i, j + 1)\} | 1 \leq i \leq m; 1 \leq j \leq n - 1\} \cup \{(i, n), \text{null}\} | 1 \leq i \leq m\}$$

$$s_2 = \{(i, j), (i + j, j)\} | 1 \leq i \leq m - 1; 1 \leq j \leq n\} \cup \{(j, m), \text{null}\} | 1 \leq j \leq n\}$$

$$\text{root} = (1, 1)$$

The chapter [12] uses the term pictures for grids. It is easy to see that over the domain of grids, regular graph constraint are equivalent to a domino system with $s_1$ edges denoting horizontal dominoes and $s_2$ edges denoting vertical dominoes. The graph isomorphism corresponds to the use of projection.

[12] states the equivalence of domino systems over pictures with negation-free regular expressions with projections, on-line tessellation automata, existential monadic second-order formulas and tiling systems.

We view the fact that, over the grids, regular graph constraints are equivalent to each of the systems above as an indication that the definition of regular graph constraints is natural.

Note When comparing our regular graph constraints to trees and domino systems, we notice that in our definition of a model there are no fixed labels associated with nodes. The only labeling of nodes comes from the graph homomorphism, which corresponds to projection in tree and picture languages. Our simplification makes our undecidability result strictly stronger. Furthermore, regular graph constraints can capture the distinction between a node with an edge terminating at null and a node with an edge terminating at a node that is not null. This distinction can be used for encoding in the structure of the graphs any fixed labeling of graph nodes.

2.7 Decidability of Implication over Graphs

Satisfiability problem over the class of graphs is trivial. Namely, $G \rightarrow G$, so every graph is satisfiable. The implication problem over graphs is also decidable, in contrast to the implication problem over the class of heaps, which we will show undecidable in Section 3.

Proposition 22

$$G_1 \sim_G G_2 \iff G_1 \rightarrow G_2$$

Proof. Let $G_1 \sim_G G_2$. Because $G_1 \rightarrow G_1$, we obtain $G_1 \rightarrow G_2$. Conversely, let $G_1 \rightarrow G_2$. By Proposition 12 we conclude $G_1 \sim_G G_2$. ■

Our regular graph constraints are weaker than finite graph acceptors of [26]. It is easy to see that finite graph acceptors can define the gridness property. Therefore, domino system satisfiability is reducible to satisfiability of finite graph acceptors, which makes finite graph acceptor satisfiability undecidable.
GraphCleanup: Repeat the following operations until the graph stabilizes:
1. remove an unreachable node
2. remove a node \( x \) such that \( s_1([x]) = \emptyset \) or \( s_2([x]) = \emptyset \)

mark(\( x \)):
1. if \( x \) is marked then return, otherwise:
2. select(\( x \))
3. pick a \( s_1 \)-successor \( y \) of \( x \); select(\( (x, y) \)); mark(\( y \))
4. pick a \( s_2 \)-successor \( z \) of \( x \); select(\( (x, y) \)); mark(\( z \))

SatisfiabilityCheck: Repeat the following operations until the graph stabilizes:
1. perform GraphCleanup;
2. if the resulting graph is empty, then \( G' \) is unsatisfiable;
3. otherwise a heap satisfying \( G' \) can be obtained as follows:
4. let all graph nodes be unmarked;
5. mark(root);
6. return subgraph containing selected nodes

Figure 1: Satisfiability check for Heaps

2.8 Satisfiability over Heaps

We show that satisfiability for heaps is efficiently decidable by the nondeterministic algorithm in Figure 1. The goal of the algorithm is to find, given a graph \( G' \), whether there exists a heap \( G \) such that \( G \rightarrow G' \). Recall that the property of a heap is that every node has exactly one \( s_1 \) outgoing edge and exactly one \( s_2 \) outgoing edge. This property need not be satisfied by \( G' \), so we cannot take \( G = G' \) to be the heap proving satisfiability of \( G' \). The algorithm updates the current graph until it becomes a heap or an empty graph. (For the purpose of this algorithm we allow even null and root to be removed from the graph.) If nonempty, the result is a heap \( G \) such that \( G \rightarrow G' \).

Proposition 23 The procedure in Figure 1 is a correct algorithm for determining satisfiability of a graph over the class of heaps.

Proof. The procedure consists of two parts: GraphCleanup and SatisfiabilityCheck. The GraphCleanup part eliminates useless nodes and determines whether there exists a heap \( H \) such that \( H \rightarrow G' \). Graph Cleanup terminates because it decreases the size of the current graph \( G \) in every step. The mark phase terminates because it does a simple breath-first search.

Observe that GraphCleanup does not reduce the set of heaps homomorphic to \( G' \). Namely, if a node \( x \) of \( G \) is removed in GraphCleanup, then no node is mapped to \( x \) under any homomorphism \( h \). Therefore, if GraphCleanup returns an empty graph, then \( G' \) is unsatisfiable.

Assume that GraphCleanup returns a nonempty graph \( G \). Then \( G \) contains root and every node in \( x \) has a \( s_1 \)-successor and a \( s_2 \)-successor, but some of the nodes may have two \( s_1 \)-successors or \( s_2 \)-successors. Invoking mark will do a depth-first search on \( G \) and pick a subgraph \( H \) where every node has exactly one \( s_1 \)-successor and one \( s_2 \)-successor. The resulting graph \( H \) will therefore be a heap. We have \( H \rightarrow G' \) because \( H \) is a subgraph of \( G' \).

2.9 Closure Properties

In this section we give a construction for computing the conjunction of two graphs and a construction for computing the disjunction of two graphs. We will use these constructions in Section 3.

2.9.1 Conjunction

We show how to use a Cartesian product construction to obtain a conjunction of two graphs \( G_1 \) and \( G_2 \).

Definition 24 (Cartesian Product) Let

\[
G^1 = (V^1, s^1_1, s^1_2, \text{null}^1, \text{root}^1) \\
G^2 = (V^2, s^2_1, s^2_2, \text{null}^2, \text{root}^2)
\]

be graphs. Then \( G^0 = G^1 \times G^2 \) is the graph

\[
G^0 = (V^0, s^0_1, s^0_2, \text{null}^0, \text{root}^0)
\]

such that:

\[
\text{null}^0 = \{\text{null}^1, \text{null}^2\} \\
\text{root}^0 = \{\text{root}^1, \text{root}^2\} \\
V^0 = \{\text{null}^0, \text{root}^0\} \\
\cup (V^1 \setminus \{\text{null}^1, \text{root}^1\}) \times (V^2 \setminus \{\text{null}^2, \text{root}^2\})
\]

\[
s^0_i = \{(x^1, x^2) \mid (x^1, y^1) \in s^1_i; (x^2, y^2) \in s^2_i, i \in \{1, 2\}\}
\]

Proposition 25 (Conjunction via Product) For every graph \( G \),

\[
G \rightarrow G_1 \times G_2 \iff G \rightarrow G_1 \text{ and } G \rightarrow G_2
\]

In other words, \( G \times G_1 \times G_2 \) is a conjunction of \( G_1 \) and \( G_2 \).

Proof. (\( \Rightarrow \)) Let \( G \rightarrow G^1 \) and \( G \rightarrow G^2 \) with \( h^1 : V \rightarrow V^1 \) and \( h^2 : V \rightarrow V^2 \) where \( G^1 \) and \( G^2 \) are defined as in Definition 24 and

\[
G = \langle V, s^1, s^2, \text{null}, \text{root} \rangle
\]

Let \( h^0 = h^1 \times h^2 \) where

\[
(h^1 \times h^2)(x) = (h^1(x), h^2(x))
\]

We claim that \( h^0 \) is a homomorphism from \( G \) to \( G^0 = G^1 \times G^2 \). It is straightforward to verify that properties 2 and 3 of homomorphism hold for \( h^0 \). Let \( i \in \{1, 2\} \). If \( (x, y) \in s_i \) then \( (h^1(x), h^1(y)) \in s^1_i \) and \( (h^2(x), h^2(y)) \in s^2_i \). Because \( h^1 \) and \( h^2 \) satisfy properties 2 and 3 of homomorphism, we have \( (h^1(x), h^1(y)), (h^2(x), h^2(y)) \in V^0 \) regardless of whether \( x, y \in \{\text{null, root}\} \) or not. We therefore conclude \( (h^1(x), h^2(x)), (h^1(y), h^2(y)) \in s^0_i \) by the definition of \( s^0 \). Hence, \( h^0 \) is a homomorphism and \( G \rightarrow G^0 \).
(⇐): Let \( G \to G^0 \) with \( h^0 : V \to V^0 \). Define \( h^1 = \pi_1 \circ h \) and \( h^2 = \pi_2 \circ h \). Let’s show that \( h^1 \) is a homomorphism (an analogous argument holds for \( h_2 \)). It is straightforward to see that properties 2 and 3 hold for \( h_1 \). For property 1, let \( i \in \{1,2\} \) and let \( (x, y) \in s_i \). Let \( h^0(x) = (x^1, x^2) \) and \( h^0(y) = (y^1, y^2) \). Because \( h^0 \) is a homomorphism, we have
\[
\langle (x^1, x^2), (y^1, y^2) \rangle \in s_i^0
\]
By definition of \( s_i^0 \) we conclude \( (x^1, y^2) \in s_i^1 \) which by definition of \( h^1 \) means \( (h^1(x), h^1(y)) \in s_i^1 \). □

2.9.2 Disjunction

Given our definition of graphs, there is no construction that would yield disjunction of arbitrary graphs over the family that contains all heaps. We illustrate this fact with an example. We then give a simple condition on graphs that ensures that the disjunction construction is possible over the domain of heaps.

Example 26 Let
\[
G^1 = \langle \{\text{root, null}\}, s^1_1, s^2_2, \text{root, null} \rangle
\]
\[
s^1_1 = \{\text{root, null}\}
\]
\[
s^1_2 = \{\text{root, root}\}
\]
and
\[
G^2 = \langle \{\text{root, null}\}, s^1_1, s^2_2, \text{root, null} \rangle
\]
\[
s^2_1 = \{\text{root, root}\}
\]
\[
s^2_2 = \{\text{root, null}\}
\]
In the class of heaps, the only model for \( G^1 \) is \( G^1 \) itself, and the only model of \( G^2 \) is \( G^2 \). Now assume that there exists graph
\[
G^0 = \langle V^0, s^0_1, s^0_2, \text{null, root} \rangle
\]
such that \( G^1 \to G^0 \) and \( G^2 \to G^0 \). From \( G^1 \to G^0 \) we conclude
\[
\langle \text{root, null} \rangle \in s^0_1
\]
and from \( G^2 \to G^0 \) we conclude
\[
\langle \text{root, null} \rangle \in s^0_2
\]
Therefore for the graph
\[
G^3 = \langle \{\text{root, null}\}, s^3_1, s^3_2, \text{root, null} \rangle
\]
\[
s^3_1 = \{\text{root, null}\}
\]
\[
s^3_2 = \{\text{root, null}\}
\]
we have \( G_3 \to G^0 \) as well. So there is no graph \( G^0 \) such that for all heaps \( G \),
\[
(G \to G^0) \ \text{iff} \ \langle (G \to G^1) \ \text{or} \ (G \to G^2) \rangle
\]
To ensure that we can find union graphs over the set of heaps, we will require \( s_2(\text{root}) = \text{null} \).

Definition 27 (Orable Graphs) A graph
\[
G = \langle V, s_1, s_2, \text{null, root} \rangle
\]
is orable iff for all \( x \in V \),
\[
\langle \text{root, } x \rangle \in s_2 \ \text{iff} \ x = \text{null}
\]

Definition 28 (Graph Sum) Let
\[
G^1 = \langle V^1, s^1_1, s^1_2, \text{null, root} \rangle
\]
\[
G^2 = \langle V^2, s^2_1, s^2_2, \text{null, root} \rangle
\]
be orable graphs such that \( V^1 \cap V^2 = \{\text{null, root}\} \). Then \( G^0 = G^1 + G^2 \) is the graph
\[
G^0 = \langle V^0, s^0_1, s^0_2, \text{null, root} \rangle
\]
where
\[
V^0 = V^1 \cup V^2
\]
\[
s^0_1 = s^1_1 \cup s^2_1
\]
\[
s^0_2 = s^1_2 \cup s^2_2
\]
The following simple fact allows us to form arbitrary finite sums of orable graphs.

Proposition 29 If \( G^1 \) and \( G^2 \) are orable graphs, then \( G^1 + G^2 \) is also orable.

Proposition 30 (Disjunction via Sum) Let \( G \) be a heap and \( G^1 \) and \( G^2 \) be orable graphs. Then
\[
G \to G^1 + G^2 \ \text{iff} \ G \to G^1 \ \text{or} \ G \to G^2
\]
Proof. (⇐): Assume without loss of generality \( G \to G^1 \). Because \( G^1 \) is a subgraph of \( G^1 + G^2 \), there exists an identity homomorphism from \( G^1 \) into \( G^1 + G^2 \). By Proposition 9 we conclude \( G \to G^1 + G^2 \). By \( \Rightarrow \): Let \( G^0 = G^1 + G^2 \) where \( G^1 \) and \( G^2 \) are as defined in Definition 28, and \( G \to G^0 \) with a homomorphism \( h : V \to V^0 \) where
\[
G = \langle V, s_1, s_2, \text{null, root} \rangle
\]
We claim
\[
h[V] \subseteq V^1 \ \text{or} \ h[V] \subseteq V^2
\]
Suppose the claim does not hold. Then there exist \( x^0, y^0 \in V \) such that
\[
x^0 \in V^1 \setminus V^2
\]
\[
y^0 \in V^2 \setminus V^1
\]
where \( x^1 = h(x^0) \) and \( y^2 = h(y^0) \). By definition of heap, there exists a sequence of nodes \( p = \text{root}, s^0_1, \ldots, x^0 \) forming a path from root to \( x^0 \) in \( G \). Because \( x^0 \notin \{\text{null, root}\} \), the path has length at least two and \( z^0 \notin \{\text{null, root}\} \) (it may or may not be \( x^0 = z^0 \)). Because \( G^1 + G^2 \) is orable, the edge from root to \( z^0 \) cannot be from \( s_2 \), so \( s_1(x^0) = z^0 \). We claim \( h(z^0) \in V^1 \setminus V^2 \). Indeed, suppose \( h(x^0) \in V^2 \). By the properties of homomorphism and because in \( G^1 + G^2 \) there are no edges from nodes \( V^1 \setminus \{\text{null, root}\} \) to nodes \( V^2 \setminus \{\text{null, root}\} \), we have \( h(w) \in V^2 \) for every node \( w \) of the

Figure 2: Graph Sum
path $p$. This is a contradiction with $h(x^0) \in V^1 \setminus V^2$. We conclude

$$z^0 \in V^1 \setminus V^2$$

Repeating an analogous argument for node $y^0$, we conclude

$$z^0 \in V^2 \setminus V^1$$

because $z^0 = s_1(\text{root})$ is the unique $s_1$-successor of root in the heap $G$. We have arrived at contradiction, so (3) is true. If $h[V] \subseteq V^1$ then $G \to G^1$ and if $h[V] \subseteq V^2$ then $G \to G^2$.

A product of orbable graphs is orbable.

**Proposition 31** Let $G^1$ and $G^2$ be orbable graphs. Then $G^1 \times G^2$ is orbable.

In the sequel we will deal only with orbable graphs.

## 3 Undecidability of Implication

This section presents the central result of this paper: The implication of graphs is undecidable over the class of heaps. Our proof proceeds in two steps. We first introduce a family of corresponder graphs. We show that satisfiability of graphs over the family of corresponder graphs is undecidable.

In the second step we show that satisfiability over corresponder graphs can be reduced the question of whether an implication between two graphs fails to hold. The key to the construction in the second step is that a conjunction of certain regular graph constraints and negations of regular graph constraints can precisely characterize the class of corresponder graphs.

### 3.1 Corresponder Graphs

Corresponder graphs are a subclass of the class of heaps. Figure 3 shows an example corresponder graph.

**Definition 32** Let $k \geq 2$, $n \geq 2$, and

\[
0 = u_0 < u_1 < \ldots < u_{k-1} < n
\]

A corresponder graph

$$CG(n, k, u_1, \ldots, u_{k-1}, l_1, \ldots, l_{k-1})$$

is a graph isomorphic to

$$G = \langle V, s_1, s_2, \text{null, root} \rangle$$

where

\[
V = \{ \text{null, root} \} \\
\cup \{ C_0, C_1, \ldots, C_{2k-1} \} \\
\cup \{ U_0, U_1, \ldots, U_{2n-1} \} \\
\cup \{ L_0, L_1, \ldots, L_{2n-1} \} \\
\cup \{ \text{root, C0} \} \\
\cup \{ \{ C_i, C_{i+1} \} \mid 0 \leq i < 2k-1 \} \\
\cup \{ \{ C_{2k-1}, \text{null} \} \} \\
\cup \{ \{ U_i, U_{i+1} \} \mid 0 \leq i < 2n-1 \} \\
\cup \{ \{ U_{2n-1}, \text{null} \} \} \\
\cup \{ \{ L_i, L_{i+1} \} \mid 0 \leq i < 2n-1 \} \\
\cup \{ \{ L_{2n-1}, \text{null} \} \}
\]

\[
s_1 = \{ \{ \text{null, root} \} \}
\]

\[
\cup \{ \{ C_i, C_{i+1} \} \mid 0 \leq i < 2k-1 \}
\]

\[
\cup \{ \{ C_{2k-1}, \text{null} \} \}
\]

\[
\cup \{ \{ U_i, U_{i+1} \} \mid 0 \leq i < 2n-1 \}
\]

\[
\cup \{ \{ U_{2n-1}, \text{null} \} \}
\]

\[
\cup \{ \{ L_i, L_{i+1} \} \mid 0 \leq i < 2n-1 \}
\]

\[
\cup \{ \{ L_{2n-1}, \text{null} \} \}
\]

The family $CG$ is the union of all corresponder graphs

$$CG(n, k, u_1, \ldots, u_{k-1}, l_1, \ldots, l_{k-1})$$

### 3.2 Corresponder Graph Satisfiability

For completeness we define the Post correspondence problem, PCP ([22], pp183).

**Definition 33** A PCP instance is a sequence of pairs of nonempty words:

$$\langle v_0, w_0 \rangle, \langle v_1, w_1 \rangle, \ldots, \langle v_m-1, w_m-1 \rangle$$

A solution for a PCP instance is a sequence

$$t_0, t_1, \ldots, t_{k-1}$$

such that

$$u_0 v_1 \ldots v_{k-1} = u_0 w_1 \ldots w_{k-1}$$

[22] contains the proof for the following theorem.

**Theorem 34** The following problem is undecidable: given a PCP instance, does it have a solution.

We will use the following proposition to establish undecidability of graph implication.

**Proposition 35** Satisfiability of graphs over the class of corresponder graphs is undecidable.

**Proof.** We give a reduction from PCP. Let $m \geq 2$ and let

$$\langle v_0, w_0 \rangle, \langle v_1, w_1 \rangle, \ldots, \langle v_m-1, w_m-1 \rangle$$

be an instance of PCP where $v_i, w_i$ are nonempty words. Introduce names $a_i^0$ and $b_i^1$ for letters in words $v_i, w_i$:

$$v_i = v_i^0 v_i^1 \ldots v_i^{m-1} \quad 0 \leq i \leq m-1$$

$$w_i = w_i^0 w_i^1 \ldots w_i^{m-1} \quad 0 \leq i \leq m-1$$

where $p_i = |v_i|$ and $q_i = |w_i|$. We construct a graph $G$ such that there exists a corresponder graph $G_0$ with the property $G_0 \to G$ iff the PCP instance has a solution. Figure 4 illustrates how a corresponder graph $G_0$ with a homomorphism from $G_0$ to $G$ encodes a solution of the PCP instance $\langle c, bc \rangle, \langle ab, a \rangle$.

Let

$$G = \langle V, s_1, s_2, \text{null, root} \rangle$$
Figure 3: An Example Corresponder Graph \((k = 2, n = 3)\)

Figure 4: Corresponder Graph with a Homomorphism Encoding a Solution of \(\langle c, bc \rangle, \langle ab, a \rangle\)
Define the components of $G$ as follows. For every pair of words $(v_i, w_i)$ introduce two nodes $c_{2i}, c_{2i+1} \in V$. These nodes will summarize $C$-nodes of a corresponder graph. For every position $v_i$ of the word $v_i$ introduce nodes $a_{2i,0}^{v_i}$ and $a_{2i+1,0}^{v_i}$ and for every position $w_i$ introduce nodes $b_{2i,0}^{w_i}$ and $b_{2i+1,0}^{w_i}$. The $\alpha$-nodes will summarize $U$-nodes and the $b$-nodes will summarize the $L$-nodes of the corresponder graph. Introduce also the additional nodes $b_{2i}^{v_i}$ to encode the information that $a_{2i,0}^{v_i}$ node has an incoming edge from a $c$-node. As we will see below, the $b_{2i}^{v_i}$ nodes have $s_2$ pointing to root as opposed to null, which ensures that every $a_{2i,0}^{v_i}$ node has an incoming edge from a $c$-node. For analogous reasons we introduce $a_{2i+1,1}^{v_i}$ nodes. Let

$$V = \{ \text{null, root} \}$$

$$\cup \{ c_0, c_1, \ldots, c_{2m-1} \}$$

$$\cup \{ a_{2i,0}^{v_i} \mid 0 \leq i < m; 0 \leq j < 2p_i \}$$

$$\cup \{ b_{2i,0}^{w_i} \mid 0 \leq i < m; 0 \leq j < 2q_i \}$$

$$\cup \{ b_{2i+1,0}^{w_i} \mid 0 \leq i < m; 0 \leq j < q_i \}$$

$$\cup \{ a_{2i+1,1}^{v_i} \mid 0 \leq i < m; 0 \leq j < p_i \}$$

Define $s_1$ graph edges as follows.

The $c$-nodes are connected into a list that begins with root and every $c_{2i}$ is followed by $c_{2i+1}$. The pairs $c_{2i}, c_{2i+1}$ for different $i$ can repeat in the list any number of times and in arbitrary order. This list will encode a PCP instance solution.

The $\alpha$-nodes representing word positions are linked in the order in which they appear in the word. The last position in a word can be followed by the first position of any other word, or by null. The nodes for the $v_i$ words and the nodes for the $w_i$ words form disjoint lists along the $s_1$ edges.

$$s_1 = \{ \text{root, } c_{2i} \mid 0 \leq i < m \}$$

$$\cup \{ \{ c_{2i}, c_{2i+1} \} \mid 0 \leq i < m \}$$

$$\cup \{ \{ c_{2i+1}, c_{2i} \} \mid 0 \leq i, j < m \}$$

$$\cup \{ \{ c_{2i+1}, \text{null} \} \mid 0 \leq i < m \}$$

$$\cup \{ \{ a_{2i,0}^{2,0}, a_{2i+1,0}^{2,1} \} \mid 0 \leq i < m; 0 \leq j < p_i; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ a_{2i+1,0}^{2,1}, a_{2i+2,0}^{2,0} \} \mid 0 \leq i < m; 0 \leq j < q_i - 1; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ a_{2i,0}^{2,0}, a_{2i+1,0}^{2,1} \} \mid 0 \leq i, j < m; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ a_{2i+1,0}^{2,1}, \text{null} \} \mid 0 \leq i < m; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ b_{2i,0}^{2,0}, b_{2i+1,0}^{2,1} \} \mid 0 \leq i < m; 0 \leq j < q_i; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ b_{2i+1,0}^{2,1}, b_{2i+2,0}^{2,0} \} \mid 0 \leq i < m; 0 \leq j < q_i - 1; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ b_{2i,0}^{2,0}, b_{2i+1,0}^{2,1} \} \mid 0 \leq i, j < m; \alpha \in \{ 0, 1 \} \}$$

$$\cup \{ \{ b_{2i+1,0}^{2,1}, \text{null} \} \mid 0 \leq i < m \}$$

We define $s_2$ graph edges as follows.

Every $c_{2i}$ edge points to the position at the beginning of the word. Every numbered node points to the $\alpha$-positions; odd numbered nodes point to $b$-positions.

The $a_i$ and $b_j$ word positions are connected so that an $\alpha$-node points to a $b$-node for even indices, whereas a $b$-node points to an $\alpha$-node for odd indices.

$$s_2 = \{ \{ \text{root, null} \} \}$$

$$\cup \{ \{ c_{2i}, a_{2i,0}^{v_i} \} \mid 0 \leq i < m \}$$

$$\cup \{ \{ c_{2i+1}, b_{2i,0}^{w_i} \} \mid 0 \leq i < m \}$$

$$\cup \{ \{ a_{2i,0}^{v_i}, b_{2i+1,0}^{w_i} \} \mid 0 \leq i, k < m; 0 \leq l < q_k; v_i = w_k^l \}$$

$$\cup \{ \{ a_{2i,0}^{v_i}, b_{2i+1,0}^{w_i} \} \mid 0 \leq i, k < m; 0 \leq j < p_i; 0 \leq l < q_k; v_i^j = w_k^l \}$$

$$\cup \{ \{ b_{2i,0}^{w_i}, \text{null} \} \mid 0 \leq k < m; 0 \leq l < q_k \}$$

$$\cup \{ \{ b_{2i+1,0}^{w_i}, \text{root} \} \mid 0 \leq k < m; 0 \leq l < q_k \}$$

$$\cup \{ \{ b_{2i+1,0}^{w_i}, v_i \} \mid 0 \leq k < m; 0 \leq j < p_i \}$$

$$\cup \{ \{ a_{2i+1,1}^{v_i}, \text{root} \} \mid 0 \leq i < m; 0 \leq j < p_i \}$$

This completes the definition of $G$.

**Claim 36** The PCP instance has a solution iff there exists a corresponder graph $G_0$ such that $G \rightarrow G_0$.

$(\Rightarrow)$: Assume that the PCP instance has a solution $t_0, t_1, \ldots, t_{k-1}$. Then

$$v_{t_0} \cdot v_{t_1} \cdots v_{t_{k-1}} = w_{t_0} \cdot w_{t_1} \cdots w_{t_{k-1}}$$

Let $u_0 = v_0 = 0$,

$$n = \sum_{j=0}^{k-1} \mid v_j \mid$$

$$u_{i+1} = \sum_{j=0}^{i} \mid v_j \mid, \quad 0 \leq i < k - 1$$

$$l_{i+1} = \sum_{j=0}^{i} \mid w_j \mid, \quad 0 \leq i < k - 1$$

and let

$$G_0 = CG(n, k, u_1, \ldots, u_{k-1}, l_1, \ldots, l_{k-1})$$

be a corresponder graph. We construct a homomorphism $h$ from $G_0$ to $G$ as follows. We map $C$-nodes of $G_0$ into $c$-nodes of $G$:

$$h(C_{2i}) = c_{2i+j}, \quad 0 \leq j < k$$

$$h(C_{2i+1}) = c_{2i+j+1}, \quad 0 \leq j < k$$

For $0 \leq f < n$, let $d_0(f)$ denote the largest index $i$ such that $u_i \leq f$. We map the $U$-nodes into $\alpha$-nodes as follows. Consider a node $U_{2f}$. Let $i = d_0(f)$. Then $U_{2f}$ is the even node that represents the letter $v_i^{j-2}$ of the word $v_i$:

$$h(U_{2f}) = a_{2i}^{j-2}, \quad i = d_0(f), \quad 0 \leq f < n$$
The mapping of $U_{2f+1}$ is similar. In this case we also encode the information whether $L_{2f+1}$ has an $s_2$-edge from a $C$-node.

$$U_{2f+1} = a_i^{(2f-u_i+1,0)}, \quad i = d_u(f), \quad 0 \leq f < n$$

where

$$\alpha_i(f) = \begin{cases} 
1, & f \in \{l_0, l_1, \ldots, l_{k-1}\} \\
0, & \text{otherwise}
\end{cases}$$

The mapping of $L$-nodes is analogous. Let $d_l(f)$ denote the largest index $i$ such that $l_i \leq f$. Then

$$L_{2f+1} = b_i^{(2f-u_i+1,0)}, \quad i = d_u(f), \quad 0 \leq f < n$$

$$U_{2f} = b_i^{(2f-u_i+1,0)}, \quad i = d_u(f), \quad 0 \leq f < n$$

where

$$\alpha_u(f) = \begin{cases} 
1, & f \in \{u_0, u_1, \ldots, u_{k-1}\} \\
0, & \text{otherwise}
\end{cases}$$

It is straightforward to verify that $h$ is indeed a homomorphism.

\[ (\Leftarrow) : \text{Assume that } G_0 \rightarrow G \text{ where } G_0 = C(n, k, u_1, \ldots, u_{k-1}, l_1, \ldots, l_{k-1}) \]

is a correspondent graph and $h$ is a homomorphism from $G_0$ to $G$. Because in graph $G$ all paths given by the regular expression $1^*$ lead to $a_i$-nodes or null, we conclude that each $C_j$ node is mapped to some $c_i$ node. For $0 \leq j < k$ we define

$$t_j = i \iff h(C_{2j}) = c_{2i}$$

From the properties of homomorphism we derive

$$t_j = i \iff h(C_{2j+1}) = c_{2i+1}$$

We will show that $t_j$ is a solution of the PCP instance. Let $u_0 = l_0 = 0$ and $u_{n+1} = l_{n+1} = n$. Let

$$\tau(a_i^{2j,0}) = \tau(a_i^{2j+1,0}) = v_i^j$$

$$\tau(b_i^{2j,0}) = \tau(b_i^{2j+1,0}) = w_i^j$$

and $h' = \tau \circ h$. By construction of $s_2$ in $G$ we have

$$h'(U_{2j}) = h'(U_{2j+1}) = h'(L_{2j}) = h'(L_{2j+1})$$

for $0 \leq j < n$. To prove

$$v_0 u_1 \ldots u_{n-1} = w_0 w_1 \ldots w_{n-1}$$

it therefore suffices to show

$$v_j = h'(U_{2u_j})h'(U_{2(u_j+1)}) \ldots h'(U_{2(u_{j+1}-1)})$$

$$w_j = h'(L_{2u_j})h'(L_{2(u_j+1)}) \ldots h'(L_{2(u_{j+1}-1)})$$

for $0 \leq j < k$. Let $t_j = i$. Then $h(C_{2j}) = c_{2i}$. We have $\langle C_{2j}, U_{2u_j} \rangle \in s_2$ in the correspondent graph $G_0$. On the other hand, $\langle c_{2i}, a_i^{0,0} \rangle \in s_2$ is the only $s_2$-outgoing edge of $c_{2i}$ in $G$. Therefore, $h(U_{2u_j}) = a_i^{0,0}$. From this, we first conclude so $h'(U_{2u_j}) = v_i^0$. Next, by construction of $s_1$ edges of $G$ and $G_0$ we get

$$h(U_{2(u_j+1)}) = a_i^{2,0}$$

$$\ldots$$

$$h(U_{2(u_j+p_i-1)}) = a_i^{(p_i-1,0)}$$

To establish (4) it suffices to show $u_j + p_i = u_{j+1}$. To see that the equality holds, suppose first $u_j + p_i > u_{j+1}$. Then $h(U_{2u_{j+1}}) = a_i^{2f,0}$ where $f > 0$. Because $h$ is a homomorphism, following $s_2$ edge once we conclude $h(L_{2u_{j+1}}) = b_i^{2f,0}$ for some $g$ and following $s_2$ for the second time we obtain the contradiction because in correspondent graph $s_2(L_{2u_{j+1}}) = \text{root}$ but in $G$ we have $s_2(b_i^{2f,0}) = \text{null}$. Similarly, suppose now $u_j + p_i < u_{j+1}$. Then $s_1(U_{2(u_j+p_i)-1})$ is not null, so let $U_{2(u_j+p_i)} = s_1(s_1(U_{2(u_j+p_i)-1}))$. Because $u_j + p_i \notin \{u_0, \ldots, u_{k-1}\}$, we get

$$s_2(s_2(U_{2(u_j+p_i)})) = \text{null}$$

On the other hand, $h(U_{2(u_j+p_i)}) = a_i^{0,0}$ for some $f$, so

$$s_2(s_2(h(U_{2(u_j+p_i)}))) = \text{root}$$

which is again a contradiction. Therefore $u_j + p_i = u_{j+1}$. Showing (5) is analogous. We conclude that $t_0, \ldots, t_{k-1}$ is a solution for PCP instance.

Our claim is therefore true and satisfiability over correspondent graphs is undecidable.

### 3.3 Defining Correspondent Graphs

In this section construct graphs $P$ and $Q$ such that

$$G_0 \rightarrow (P \land \neg Q)$$

iff $G_0$ is a correspondent graph.

When presenting the graphs $P, Q_0, \ldots, Q_{16}$ we use the following conventions. We use the label $r$ to denote the root of the graph. We label the edges of the relation $s_1$ relation by $1$ and the edges of $s_2$ by $2$. Note that if a node has no outgoing edges, it would be useless in the graph in terms of specifying a set of models $G_0$. Every graph node will therefore have at least one outgoing edge for every label. However, in order to make the graph sketches clearer, if a node $x$ has an outgoing edge with label $a$ to every node in the graph, we will simply omit all $a$ edges of node $x$ from the sketch. In particular, if a node has no outgoing edges in the graph sketch, it means that its outgoing edges are unconstrained. A double-headed arrow from node $x$ to node $y$ with label $a$ denotes two single arrows, one from $x$ to $y$ and one from $y$ to $x$, both labeled with $a$. We do not show the edge (root, null) $\in s_2$ that is always present in an orbable graph. We also do not show the edges originating from null. We will be free to display null several times in the same picture, all these occurrences denote to the unique root node in the graph.

The graph $P$ in Figure 5 is our first approximation of a correspondent graph. Unfortunately, $P$ allows some models that are not correspondent graph, such as the example in Figure 6. This is why we introduce the graph $Q$. The graph $Q$ appears in (6) negated and we design it to contain models of $P$ that are not correspondent graphs.

![Figure 7: Graph $Q_0$](image-url)
Figure 5: Graph $P$

Figure 6: A model of $P$ that is not a corresponder graph
We construct $Q$ as a sum of orable graphs:

$$Q = Q_0 + Q_1 + \cdots + Q_{16}$$

The idea behind the construction of these graphs comes from the proof of Proposition 37; we now give only an informal overview of the graphs. The graphs $Q_0$ (Figure 7), $Q_1$ (Figure 8), $Q_2$ (Figure 9), and $Q_3$ (Figure 10) eliminate certain cycles from the set of models of $P$. The graphs $Q_4$ (Figure 11), $Q_5$ (Figure 12), $Q_6$ (Figure 13), and $Q_7$ (Figure 16) ensure that different paths in the graph lead to the same object. The graphs $Q_8$ (Figure 14) and $Q_9$ (Figure 15) ensure that there is the same number of $U$ and $L$-nodes in a model of $P$. The graphs $Q_{10}$ (Figure 17) and $Q_{11}$ (Figure 18) ensure that $U$ or $L$ nodes have an $s_2$ edge to root if $U$ or $L$ node in the same column has an $s_2$-edge from a $C$-node. The graphs $Q_{12}$ (Figure 19) and $Q_{13}$ (Figure 20) ensure that a $C$-node that is later in the $C$-list has an edge to a node that is later in the $U$ or $L$ list. Finally, graphs $Q_{14}$ (Figure 21), $Q_{15}$ (Figure 22) and $Q_{16}$ (Figure 23) ensure that $C$-nodes have $s_2$ edges only to $U$ and $L$-nodes, and that an $L$ or $U$ node can only have an edge to root, null, a $U$-node, or an $L$-node.

We can now show the key step in the undecidability proof for the implication of graph constraints.

**Proposition 37**

$$G_0 \rightarrow (P \land \neg Q)$$  \hspace{1cm} (7)

**Proof.** ($\Leftarrow$): Let $G_0$ be a correspond graph

$$G_0 = \mathcal{C}G(n, k, u_1, \ldots, u_{k-1}, l_1, \ldots, l_{k-1})$$
We show that \( G_0 \rightarrow P \) and for all \( 0 \leq i \leq 16 \), it is not the case that \( G_0 \rightarrow Q_i \).

\((G_0 \rightarrow P)\): Define homomorphism \( h \) from \( G_0 \) to \( P \) as follows.

\[
egin{align*}
h(C_0) &= c_0 \\
h(C_1) &= c_1 \\
h(C_{2j+2}) &= c_2, \quad 0 \leq j < k - 1 \\
h(C_{2j+3}) &= c_3, \quad 0 \leq j < k - 1 \\
h(U_0) &= a_0 \\
h(U_1) &= a_1 \\
h(U_{2j+2}) &= \begin{cases} a_2, & j + 1 \in \{u_1, \ldots, u_{h-1}\} \\ a_4, & \text{otherwise} \end{cases} \\
h(U_{2j+3}) &= \begin{cases} a_3, & j + 1 \in \{l_1, \ldots, l_{h-1}\} \\ a_5, & \text{otherwise} \end{cases} \\
h(L_0) &= b_0 \\
h(L_1) &= b_1 \\
h(L_{2j+2}) &= \begin{cases} b_2, & j + 1 \in \{u_1, \ldots, u_{h-1}\} \\ b_4, & \text{otherwise} \end{cases} \\
h(L_{2j+3}) &= \begin{cases} b_3, & j + 1 \in \{l_1, \ldots, l_{h-1}\} \\ b_6, & \text{otherwise} \end{cases}
\]

It is straightforward to verify that \( h \) is indeed a homomorphism, so \( G_0 \rightarrow P \).

\((\neg G_0 \rightarrow Q_0)\): Apply Proposition 18 with \( c = 1^* \).

\((\neg G_0 \rightarrow Q_1)\): Apply Proposition 18 with \( c = 121^* \).

\((\neg G_0 \rightarrow Q_2)\): Apply Proposition 18 with \( c = 1221^* \).

\((\neg G_0 \rightarrow Q_3)\): Apply Proposition 18 with \( c = 12(21)^* \).

\((\neg G_0 \rightarrow Q_4)\): Suppose \( h \) is a homomorphism from \( G_0 \) to \( Q_4 \). By mapping the path

\[\text{root}, 1, C_0, 2, U_0, 1, U_1\]

we conclude \( h(U_1) = a_1 \). By mapping the path

\[\text{root}, 1, C_0, 2, U_0, 2, L_0, 1, L_1, 2, U_1\]

we conclude \( h(U_1) = a_2 \), which is a contradiction.

\((\neg G_0 \rightarrow Q_5)\): Suppose \( h \) is a homomorphism from \( G_0 \) to \( Q_5 \). By mapping the slice in \( 121^* \) with \( h \) we conclude that there exists a node \( U_j \) in \( G_0 \) such that \( h(U_j) = a_0 \). Now as in the previous case we get that \( h(U_{j+1}) = a_1 \) and \( h(U_{j+2}) = a_2 \), which is a contradiction.

\((\neg G_0 \rightarrow Q_6)\): Similarly to the previous case, map the slice in \( 1221^* \) to conclude that for some node \( L_j \) we have \( h(L_j) = b_0 \) and then obtain a contradiction.

\((\neg G_0 \rightarrow Q_7)\): Similarly to the previous cases, suppose \( h \) is a homomorphism from \( G_0 \) to \( Q_7 \). By mapping the slice in \( 1221^* \) via \( h \), we conclude that there exists a node \( L_j \) in \( G_0 \) such that \( h(L_j) = b_0 \). By construction of \( Q_7 \), it must be \( s_1(L_j) = \text{null} \). Furthermore, \( h(U_j) = a_0 \) and \( s_1(U_j) \neq \text{null} \). This is a contradiction with the fact that \( G_0 \) is a correspond graph.

\((\neg G_0 \rightarrow Q_8)\): This fact is analogous to the previous one.

\((\neg G_0 \rightarrow Q_9)\): Suppose \( h \) is a homomorphism from \( G_0 \) to \( Q_9 \). As in the case \( \neg G_0 \rightarrow Q_4 \) we conclude \( h(L_1) = b_1 \) and \( h(L_1) = b_2 \), a contradiction.

\((\neg G_0 \rightarrow Q_{10})\): Suppose \( h \) is a homomorphism from \( G_0 \) to \( Q_{10} \). As in case \( \neg G_0 \rightarrow Q_5 \), for some node \( U_j \) in
\[ G_0 \text{ we have } h(U_j) = a_0. \] Next it follows \( h(L_j) = b_0 \) and
\[ s_2(L_j) = \text{root}, \] which implies \( j = 2a_r \) where \( 0 \leq r < k. \) Therefore \( s_2(C_{2r}) = U_j. \) But this is in contradiction with the fact that \( h(U_j) = a_0 \) and \( a_0 \) has no incoming \( s_2 \)-edges in \( Q_{10}. \)

\[ (\neg G_0 \rightarrow Q_{11}): \text{This case is analogous to the previous one.} \]

\[ (\neg G_0 \rightarrow Q_{12}): \text{Suppose } h \text{ is a homomorphism from } G_0 \text{ to } Q_{10}. \text{ By mapping the slice in } 1^* \text{ from } G_0 \text{ to } G \text{ we conclude that there exists a node } C_{2j} \text{ in } G_0 \text{ such that } h(C_{2j}) = c_2 \text{ and a node } C_{2i} \text{ where } i \geq j + 2 \text{ such that } h(C_{2i}) = c_0. \text{ Since } s_2(c_2) = a_0, \text{ we conclude } h(U_{2u_i}) = a_0. \] By mapping the path
\[ U_{2j}, 1, U_{2j+1}, 1, U_{2j+2}, 1, \ldots, 1, \text{null} \]
with homomorphism \( h \), we conclude that there exists be some \( U \)-node with even index that is mapped to \( a_4 \). So let \( U_{2(u_j+t)} \) be such node with the least index. Then
\[ h(U_{2(u_j+t)}) = a_4 \] and
\[ h(U_{2(u_j+t+1)}) = a_2 \]
for \( 1 \leq r < t. \] Let \( 1 \leq r < t. \) Then \( h(L_{2(u_j+t)}) = b_0 \) so
\[ s_2(L_{2(u_j+t)}) = \text{null}, \] which means that \( u_j + r < u_{j+1} \) for all \( 1 \leq r < t, \) so \( u_j + t < u_{j+1} < u_i. \) Therefore
\[ s_2(C_{2i}) \neq U_{2(u_j+r)} \]
h is a homomorphism and \( h(C_{2i}) = c_0, \) so
\[ h(U_{2u_i}) = a_4 \] (8)
The corresponder graph \( G_0 \) contains a path in \( 1^* \) from \( U_{2(u_j+t)} \) to \( U_{2u_i} \) where \( U_{2(u_j+t)} \) and \( U_{2u_i} \) are distinct nodes because \( u_j + t < u_i. \) Because \( h(U_{2(u_j+t)}) = h(U_{2u_i}) = a_4, \) there exists a cyclic \( s_1 \)-path from \( a_0 \) to \( a_1 \) in \( Q_{12}, \) a contradiction with the definition of \( Q_{12}. \)

\[ (\neg G_0 \rightarrow Q_{13}): \text{This case is analogous to the previous one.} \]

\[ (\neg G_0 \rightarrow Q_{14}): \text{Suppose } h \text{ is a homomorphism from } G_0 \text{ to } Q_{13}. \text{ Then } h(C_i) = c_2 \text{ for some } i. \] Let \( S = s_2(C_i). \text{ Then } S = U_j \text{ or } S = L_j \text{ for some } j \text{ and } h(S) = a_0. \) On the other hand, by mapping the slice 121* and 1221*, we conclude that all \( U \)-nodes are mapped to \( a_0 \) and \( a_1 \) whereas all \( L \)-nodes are mapped to \( b_0 \) and \( b_1. \) This is a contradiction with \( h(S) = a_2. \)

\[ (\neg G_0 \rightarrow Q_{15}): \text{Suppose } h \text{ is a homomorphism from } G_0 \text{ to } Q_{15}. \text{ Then for some node } L_j \text{ we have } h(L_j) = b_2 \text{ and therefore } h(U_j) = a_2. \text{ On the other hand by mapping the slice 121* we conclude that all } U \text{-nodes are mapped to } a_0 \text{ and } a_1, \text{ which is a contradiction.} \]

\[ (\neg G_0 \rightarrow Q_{16}): \text{This case is analogous to the previous one.} \]

\[ (\Rightarrow): \text{Let} \]
\[ G_0 = (V, s_1, s_2, \text{null}, \text{root}) \]
Assume that \( G_0 \rightarrow P \) and for all \( 0 \leq i \leq 16 \) it is not the case that \( G_0 \rightarrow Q_i. \) We will show that \( G_0 \) is a corresponder graph. Let
\[ C_0 = s_1(\text{root}) \]
\[ C_1 = s_1(C_0) \]
\[ C_2 = s_1(C_1) \]
\[ \ldots \]
Next, consider the sequence
\[
U_0 = s_2(C_0) \\
U_1 = s_1(U_0) \\
U_2 = s_1(U_1) \\
\ldots
\]
Because \( \neg G_0 \rightarrow Q_1 \) and \( G_0 \rightarrow P \) there must exist some \( n_1 \) such that \( U_{2n_1} = \text{null} \) and
\[
U_i \notin \{ \text{null, root} \} \cup \{ C_0, \ldots, C_{2k-1} \} \\
\cup \{ U_0, \ldots, U_{i-1} \}
\]
for \( 0 \leq i < 2n_1 \). Because \( \neg G_0 \rightarrow Q_2 \) and \( G_0 \rightarrow P \), considering the sequence
\[
L_0 = s_2(U_0) \\
L_1 = s_1(L_0) \\
L_2 = s_1(L_1) \\
\ldots
\]
we conclude there must exist \( n_2 \) such that \( L_{2n_2} = \text{null} \) and
\[
L_i \notin \{ \text{null, root} \} \cup \{ C_0, \ldots, C_{2k-1} \} \\
\cup \{ U_0, \ldots, U_{2n_1-1} \} \\
\cup \{ L_0, \ldots, L_{i-1} \}
\]
for \( 0 \leq i < 2n_2 \). Let
\[
V_0 = \{ \text{null, root} \} \cup \{ C_0, \ldots, C_{2k-1} \} \\
\cup \{ U_0, \ldots, U_{2n_1-1} \} \\
\cup \{ L_0, \ldots, L_{2n_2-1} \}
\]
Figure 24 shows the shape of the portion of \( G_0 \) identified so far. By construction,
\[
s_1[V_0] \subseteq V_0
\]
In the sequel we will show that \( s_2[V_0] \subseteq V_0 \) holds as well. By definition of heap, all nodes in \( G_0 \) are reachable from root, which will imply \( V \setminus V_0 = \emptyset \). We will also show that \( n_1 = n_2 \) and that \( G_0 \) satisfies the invariants that make it isomorphic to a correspondent graph.

We first observe that \( s_2(C_1) = L_1 \). Indeed, suppose that \( s_2(C_1) \neq L_1 \). From \( G_0 \rightarrow P \) follows \( s_2(C_1) \notin \{ \text{null, root} \} \). We can then show \( G_0 \rightarrow Q_0 \), which is a contradiction.

We now show that the \( s_2 \) edges between \( U \)-nodes and \( L \)-nodes form a \( (2 \times n) \)-grid where \( n = n_1 = n_2 \). First we observe that \( s_2(L_1) = U_1 \), otherwise we would have \( G_0 \rightarrow Q_4 \). Next we claim that every non-null \( s_2 \) edge originating from a \( U \)-node terminates at an \( L \)-node. Suppose \( s_2(U_j) \) is not an \( L \)-node. It cannot be a \( C \)-node or a \( U \)-node because \( G_0 \rightarrow P \). The only remaining possibility is that \( s_2(U_j) \) is a node outside \( V_0 \). But then \( G_0 \rightarrow Q_{16} \), a contradiction. Similarly, because \( \neg G_0 \rightarrow Q_{15} \), every non-null \( s_2 \) edge of an \( L \)-node terminates at a \( U \)-node. Finally, we claim that for all \( j \geq 0 \), either \( U_j = L_j = \text{null} \), or all of the following holds:

- \( \text{null} \notin \{ U_{2j}, L_{2j}, U_{2j+1}, L_{2j+1} \} \)
- \( s_2(U_{2j}) = L_{2j} \)
- \( s_2(L_{2j+1}) = U_{2j+1} \)

We have already established the claim for \( j = 0 \). Suppose the claim does not hold for all \( j \). Consider the least \( j > 0 \) for which the claim does not hold. Then one of the nodes \( U_{2j}, L_{2j} \) is null. Assume \( U_{2j} \neq \text{null} \) and \( L_{2j} = \text{null} \). Then \( G_0 \rightarrow Q_7 \), a contradiction. Similarly, if \( U_{2j} \neq \text{null} \) and \( L_{2j} \neq \text{null} \), then \( G_0 \rightarrow Q_8 \), again a contradiction. So \( U_{2j} \neq \text{null} \) and \( L_{2j} \neq \text{null} \). Then from \( G_0 \rightarrow P \) follows \( U_{2j+1} \neq \text{null} \) and \( L_{2j} \neq \text{null} \). From the previous discussion and \( G_0 \rightarrow P \) we conclude
\[
s_2(U_{2j}) = L_{2j}
\]
for some \( i \geq 0 \). We want to show \( i = j \). Suppose \( i < j \). Then there is a cycle \( p \) starting at \( U_0 \) such that \( \text{word}(p) \in (21)^* \). But then \( G_0 \rightarrow Q_3 \), a contradiction. Now suppose \( i > j \). Then \( G_0 \rightarrow Q_6 \), a contradiction. Therefore \( i = j \) and \( s_2(U_{2j}) = L_{2j} \). We similarly establish \( s_2(L_{2j+1}) = U_{2j+1} \) using the fact \( G_0 \rightarrow Q_5 \). This establishes our claim for all \( j \geq 0 \). We conclude that \( n_1 = n_2 \) and \( U \) and \( L \)-nodes are linked as in Figure 25.

We next consider \( s_2 \)-edges of \( C \)-nodes and find the values \( v_1, \ldots, v_k, u_1, \ldots, u_k \). First we show that \( s_2(C_2j) \) is a \( U \)-node for \( 0 \leq j < k \). Suppose \( s_2(C_2j) \) is not a \( U \)-node. Because \( G_0 \rightarrow P \), we conclude \( s_2(C_2j) \notin V_0 \). But then \( G_0 \rightarrow Q_{14} \), a contradiction. We similarly establish from \( G_0 \rightarrow P \) and \( \neg G_0 \rightarrow Q_{14} \) that \( s_2(C_{2j+1}) \) is an \( L \)-node for \( 0 \leq j < k \). From \( G_0 \rightarrow P \) it follows that \( s_2(C_{2j}) = U_{2j} \) for some \( i \) and \( s_2(C_{2j+1}) = L_{2j+1} \) for some \( j \).

We can therefore define \( u_j \) and \( l_j \) such that
\[
s_2(C_{2j}) = U_{2uj} \\
s_2(C_{2j+1}) = U_{2uj+1}
\]
for \( 0 \leq j < k \).

We next show
\[
s_2(L_{2j}) = \text{root} \iff \exists i, s_2(C_{2i}) = U_{2j}
\]
From \( G_0 \rightarrow P \) we have \( s_2(L_{2j}) \in \{ \text{null, root} \} \). Moreover, if \( s_2(C_{2i}) = U_{2j} \), then \( s_2(L_{2j}) \neq \text{null} \). It remains to show that \( s_2(L_{2j}) = \text{root} \) implies \( s_2(C_{2i}) = U_{2j} \) for some \( i \). Suppose that \( s_2(L_{2j}) = \text{root} \) but \( \forall i, s_2(C_{2i}) \neq U_{2j} \). Then \( G_0 \rightarrow Q_{10} \), a contradiction. We similarly establish
\[
s_2(U_{2j+1}) = \text{root} \iff \exists i, s_2(C_{2i+1}) = L_{2j+1}
\]
using \( \neg G_0 \rightarrow Q_{11} \).

We claim
\[
u_j < u_{j+1}
\]
for \( 0 \leq j < k-1 \). Suppose the claim is false and let \( j \) be the smallest index for which \( u_j \leq u_{j+1} \). Let \( i \) be such that \( u_i \) is the largest among \( u_0, \ldots, u_{j+1} \) with the property \( u_i < u_{j+1} \). Clearly \( 0 \leq i < j \). Then there exists a homomorphism \( h \) from \( G_0 \rightarrow Q_{12} \) such that \( h(U_{2u_i}) = a_0 \) and \( h(U_{2u_{j+1}}) = a_4 \). This is a contradiction with \( \neg G_0 \rightarrow Q_{12} \). We similarly conclude
\[
l_j < l_{j+1}
\]
for \( 0 \leq j < k-1 \), using \( \neg Q_{13} \).

Finally we observe that we have identified all \( s_2 \)-edges from \( V_0 \), so \( s_2[V_0] \subseteq V_0 \). Therefore \( V \setminus V_0 = \emptyset \). We conclude that \( G_0 \) is isomorphic to
\[C_2(k, k, \ldots, k, k, \ldots, k, \ldots, k)\]
3.4 The Undecidability Result

**Theorem 38** The implication of graphs is undecidable over the class of heaps.

**Proof.** We will reduce satisfiability of graphs over the class of corresponder graphs to the problem of finding a counterexample to an implication of graphs over the class of heaps. Given the reduction in Proposition 35, this will establish that the implication of graphs is Turing co-recognizable and undecidable.

Let \( G \) be a graph. Consider the implication

\[
(G \times P) \sim_H Q
\]  

(10)

We claim that \( G_0 \) is a counterexample for this implication iff \( G_0 \) is a corresponder graph such that \( G_0 \rightarrow G \).

Assume that \( G_0 \) is a corresponder graph and \( G_0 \rightarrow G \). By Proposition 37, we have \( G_0 \rightarrow P \) and \( -G_0 \rightarrow Q \). We then have \( G_0 \rightarrow (G \times P) \). Since \( -G_0 \rightarrow Q \), we conclude that \( G_0 \) is a counterexample for (10).

Assume now that \( G_0 \) is a counterexample for (10). Then \( G_0 \rightarrow G \times P \) and \( -G_0 \rightarrow Q \). Since \( G_0 \rightarrow P \) and \( -G_0 \rightarrow Q \), by Proposition 37 we conclude that \( G_0 \) is a corresponder graph. Furthermore, \( G_0 \rightarrow G \). \( \blacksquare \)

3.5 Discussion

In this section we give comments on our proof of the undecidability of implication and state some implications of this result for checking properties of programs.

---

3.5.1 Graph Equivalence and Negation

**Definition 39** We say that graphs \( G_1 \) and \( G_2 \) are equivalent over the class of graphs \( C \) and write

\[
G_1 \approx_C G_2
\]

iff

\[
G_0 \rightarrow G_1 \text{ iff } G_0 \rightarrow G_2
\]

for every graph \( G_0 \in C \).

**Proposition 40** Equivalence of graphs over the class of heaps is undecidable.

**Proof.** From Proposition 25 we have

\[
G_1 \sim_H G_2
\]

iff

\[
G_1 \approx_H G_1 \times G_2
\]

The result then follows from Proposition 38. \( \blacksquare \)

We also observe that regular graph constraints over heaps are not closed under the negation. Indeed, assume that for every graph \( G \) there exists a graph \( \overline{G} \) such that the heap models of \( \overline{G} \) are all heaps that are not models of \( G \). Then finding a counterexample to an implication \( P \sim_H Q \) is reduced to satisfiability of the graph

\[
P \times \overline{G}
\]

This is a contradiction because Proposition 23 implies that satisfiability over heaps is decidable whereas Proposition 38 implies that finding a counterexample to \( P \sim_H Q \) is undecidable.
3.5.2 Implication of Acyclic Heaps

Corresponding graphs are a cyclic subclass of the class of heaps. The cyclicity, however, is not at all essential for our construction. We argue that implication of graphs is also undecidable over the class of acyclic heaps. We can define a minor variation of corresponding graphs where $U$ and $L$ nodes never point back to root. Instead, we introduce a special node different from null to indicate the difference between columns $j$ for

$$j \in \{u_1, \ldots, u_{k-1}\} \cup \{l_1, \ldots, l_{k-1}\}$$

and the remaining columns. The resulting graphs are acyclic heaps. As a result, we have the following fact.

**Proposition 41** Implication of graphs is undecidable over the class of acyclic heaps.

3.5.3 Alternative Proofs

An alternative way to prove undecidability would be to show that conjunction of regular graph constraints and their negations can express graphs similar to grids (instead of corresponding graphs). While the construction using grids may be possible, we have found the construction using corresponding graphs to be simpler. The reason is that corresponding graphs, unlike grids, are essentially one-dimensional structures.

Our proof of Proposition 37 could potentially be simplified by showing that a larger fragment of MSOL can be written in the form of negation of an implication of graphs. We consider formulas that can be reduced to checking negation of an implication between graphs. Let a literal be a formula constructed from an orable graph as in Section 2.5. Define a **homogeneous clause** as a disjunction of positive literals:

$$C_i = A_i^0 \lor \cdots \lor A_i^n$$

or a disjunction of negative literals:

$$D_i = (\neg B_i^0) \lor \cdots \lor (\neg B_i^n)$$

Then any conjunction of positive and negative clauses

$$C_0 \land \cdots \land C_{n-1} \land D_0 \land \cdots \land D_{n-1}$$

is expressible as a negation of implication of graph constraints. This fragment appears quite expressive, but we have not been able to obtain a characterization of the fragment that allows a natural encoding a subclass like grids or correspond graphs in a way simpler than in Proposition 37.

3.5.4 Consequences for Program Checking

Implication of graphs arises if procedure specifications are regular graph constraints.

**Example 42** Consider a procedure $p$ whose precondition is that the program heap is homomorphic to a graph $G_1$ and a procedure $q$ whose precondition is that the program heap is homomorphic to a graph $G_2$ (Figure 26). If the first statement in the body of $p$ is a call to $q()$, a program checker must ensure that implication $G_1 \rightarrow G_2$ holds.

```plaintext
procedure p()
    pre G1;
    q();
}
procedure q()
    pre G2
    {
        ...
}
```

Figure 26: Program Checking Requires $G_1 \rightarrow G_2$

![Figure 26](image)

Figure 27: Ensuring an invariant requires implication

We next show that the implication problem also arises when maintaining an invariant at every program point, if the invariant is a regular graph constraint. Let

$$G_1 = (V^1, s_1, s_2, \text{null}, \text{root}^1)$$

$$G_2 = (V^2, s_1, s_2, \text{null}, \text{root}^2)$$

be orable graphs such that there are no edges from nodes $V^1$ to $\text{root}^1$ and no edges from nodes $V^2$ to $\text{root}^2$. Construct the graph $G$ (Figure 27) as

$$G = (V, s_1, s_2, \text{null}, \text{root})$$

where

$$V = \{\text{root}, a, b\} \cup V^1 \cup V^2$$

$$s_1 = \{(\text{root}, a), (\text{root}, b), (a, \text{root}^1), (b, \text{root}^2)\}$$

$$s_2 = \{(\text{root}, \text{null}), (a, \text{root}), (b, \text{null})\}$$

and where

$$\{\text{root}, a, b\} \cap (V^1 \cup V^2) = \emptyset$$

Suppose that we have a program checking system that verifies that a graph constraint is true after every statement. Consider the statement

$$\text{root.1.2} := \text{null} \quad (11)$$

Let

$$G^0 = (V^0, s_1^0, s_2^0, \text{null}^0, \text{root}^0)$$

be the graph before the statement. After the statement the resulting graph is
\[ G' = (V' \cup s_1, s_2, \text{null}, \text{root'}) \]
where the value of 2-edge from \( x \) has changed so that it points to null:
\[ s_1 = s_2[x := \text{root}] \]

Our program checking system needs to verify that for all heaps \( G_0 \),
\[ (G_0 \rightarrow G) \implies (G' \rightarrow G) \quad (12) \]
Let \( h \) a homomorphism from \( G_0 \) to \( G \). Let \( x = s_0'(s_1'(\text{root})) \). Then
\[ h(x) = \text{root'}^1 \]
or
\[ h(x) = \text{root'}^2 \]
Moreover, \( \text{root'}^1 \) and \( \text{root'}^2 \) are reachable only through the path \( \text{root'}^3 \), so no nodes other than \( x \) may be mapped to \( \text{root'}^1 \) or \( \text{root'}^2 \). We can therefore show that the implication (12) is equivalent to
\[ G_1 \sim G_2 \quad (13) \]
As explained in Section 3.5.2, we can modify the construction in the proof of Proposition 37 such that \( P \) and \( Q \) have no edges terminating at \( \text{root} \). We then let \( G_1 = P \) and \( G_2 = Q \). From the undecidability of the implication of graphs over the domain of heaps it follows that maintaining an invariant expressed as a regular graph constraint is undecidable, even across a simple assignment statement such as (11).

4 Related Work

The idea of typestate as system for statically verifying changing properties of objects was proposed in [23] and extended in [24]. The original typestate system as well as the more recent work in the context object oriented programming [6] do not support constraints over dynamically allocated objects, which is the focus of our paper.

Several recent systems support tree-like dynamically allocated data structures [23, 28, 9, 18]. The restriction to tree-like data structures is in contrast to our notion of heap, which allows cycles. The presence of non-tree data structures is one of the key factors that make the implication of regular graph constraints undecidable.

The idea of representing properties of a statically unbounded number of heaps by homomorphically mapping them to a bounded family of graphs is pervasive in the work on shape analysis [16, 2, 11, 19, 20]. These analyses use abstractions that capture approximate properties of data structures even if they are not tree-like. This feature of shape analyses makes our results directly applicable. Our undecidability result implies inability to semantically check implication or equivalence of such abstractions.

Shape analysis techniques were applied to a typestate checking problem in role analysis [14]. The compositional-ity of the analysis and the presence of procedure speciﬁcations made the need for solving the implication of constraints in [14] explicit. The algorithm [14] uses “context matching” as a decidable approximation for the implication of constraints. In [13] it was suggested that the implication problem for role constraints is undecidable. The argument makes use of acyclicity constraints as well as the constraints on the number of incoming edges of a node. In the present paper, we have shown that undecidability holds even for the regular graph constraints, which cannot directly specify acyclicity or the number of incoming edges of a node. This makes the present undecidability result strictly stronger than the result in [13].

We were pleased to discover that the constraints derived as a simplification of role analysis constraints generalize the notions of tree automata [27, 3] and a whole family of equivalent systems over grids [12]. The remarkable fact that MSOL over trees is equivalent to tree automata inspired the question which classes of graphs have decidable MSOL theory [4]. In this paper we have introduced regular graph constraints which can be seen as an alternative to MSOL in generalizing projections of local properties over trees and grids. Although regular graph constraints are strictly weaker than MSOL (and in fact the satisfiability of regular graph constraints is decidable over heaps), we have shown that the implication for regular graph constraints over heaps is undecidable.

5 Conclusion

We have proposed regular graph constraints as an abstraction of mutually recursive properties of objects in potentially cyclic graphs. We presented some evidence that regular graph constraints are a natural generalization of the tree automata and domino systems. We have shown that satisfiability of regular graph constraints is decidable over the domain of heaps. As a main result, we have shown that the implication of regular graph constraints is undecidable. The consequence of this result is that verifying that procedure preconditions are satisfied as well as maintaining program invariants is undecidable if these properties are expressed as regular graph constraints.

We have seen that decidability of problems with regular constraints is sensitive to the choice of the class of graphs. In particular, a smaller class of graphs need not imply better decidability properties. This indicates that techniques for reasoning about different classes of graphs may be substantially different. We conclude that a good support for mechanized reasoning about data structures would likely contain a set of specialized reasoning techniques for different classes of graphs.

Acknowledgements We thank Chandrasekhar Boyapati, Yuri Gurevich, Patrick Lam and Andreas Podelski for useful discussions. We thank Chandrasekhar Boyapati and Patrick Lam for useful comments on a draft of this paper.

References


