

On the invertibility of the XOR of rotations of a binary word

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November 10, 2009

Abstract

We prove the following result regarding operations on a binary word whose length is a power of two: computing the exclusive-or of a number of rotated versions of the word is an invertible (one-to-one) operation if and only if the number of versions combined is odd.

(This result is not new; there is at least one earlier proof, due to Thomsen in his PhD thesis [12]. Our proof may be new.)

Keywords: invertibility, exclusive-or, rotation, binary words, circulant matrix.

1 Introduction and proof of main result

This short note considers some simple operations on binary words.

We only consider binary words whose length is a power of two, as this is typically the case for actual computer operations (e.g., with 32-bit or 64-bit words).

We focus on operations based on rotations and exclusive-ors, as these are typically standard built-in operations.

Simple invertible operations such as these are used in many applications, such pseudo-random number generation [7, 9], encryption [4], and cryptographic hash function design [10].

We state and prove the main result, and then provide some related discussion afterwards.

Theorem 1 *If n is a power of two, v is an n -bit word, and r_1, r_2, \dots, r_k are distinct fixed integers modulo n , then the function*

$$R(v) = R(v; r_1, r_2, \dots, r_k) = (v \lll r_1) \oplus (v \lll r_2) \oplus \dots \oplus (v \lll r_k) \quad (1)$$

is invertible if and only if k is odd, where $(v \lll r)$ denotes the n -bit word v rotated left by r positions, and where “ \oplus ” denotes the bit-wise “exclusive-or” of n -bit words.

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Proof: Let $V = \{0, 1\}$, and let V^n denote the set of all n -bit words. We identify V^n with $GF(2)^n$, the set of n -element vectors over the finite field $GF(2)$.

With this identification, R is a linear operation over V^n ; $R(v)$ may be obtained by multiplying v by an $n \times n$ circulant matrix over $GF(2)$ having k ones per row and per column. (An equivalent statement of our theorem is that when n is a power of two, an $n \times n$ circulant matrix over $GF(2)$ is invertible if and only if the number k of ones in each row is odd.)

We define the Hamming weight (or weight) of an n -bit word v to be the number of ones in v .

Our proof identifies words in V^n with polynomials in $GF(2)[x]$ of degree less than n .

For each n -bit word v we define an associated polynomial $v(x)$ in $GF(2)[x]$ in the natural way: if

$$v = (v_{n-1}, v_{n-2}, \dots, v_1, v_0)$$

then the associated polynomial $v(x)$ is

$$v(x) = \sum_{i=0}^{n-1} v_i x^i.$$

For example, the unit-weight word u_i having a one in position i is associated with the polynomial $u_i(x) = x^i$. This association between words and polynomials is one-to-one.

Let $f_n(x) = x^n + 1$, a polynomial in $GF(2)[x]$. We now work with polynomials modulo $f_n(x)$, so that rotation can be effected by polynomial multiplication modulo $f_n(x)$, as is typically done when working with cyclic error-correcting codes (see [6, Section 9.2]) or circulant matrices (see [1]).

Now the word

$$(v \lll r)$$

is associated with the polynomial

$$v(x) * u_r(x) \pmod{f_n(x)};$$

reducing modulo f_n captures the effects of the rotation. In other words, multiplying by $u_r(x)$ modulo $f_n(x)$ represents a left-rotation by r positions.

Computing $R(v)$ combines the effect of several rotations, so the word $R(v)$ is associated with the polynomial

$$v(x) * r(x) \pmod{f_n(x)}$$

where

$$r(x) = x^{r_1} + x^{r_2} + \dots + x^{r_k}.$$

Note that R is an invertible operation if and only if $r(x)$ is relatively prime to $f_n(x)$; (This result is due to Guan et al. [5, Theorem 2.4]; see also Bini et al. [1, Theorem 2.2].) If $\gcd(r(x), f_n(x)) = 1$, then an inverse to $r(x)$ modulo $f_n(x)$ can be found by the extended version of Euclid's algorithm, otherwise no inverse exists. These propositions hold whether or not n is a power of two.

If n is a power of two, then

$$f_n(x) = x^n + 1 = (x + 1)^n ,$$

since we are working in $GF(2)$ (see [6, Thm. 1.46]). In this case, $r(x)$ is relatively prime to $f_n(x)$ if and only if $r(x)$ is relatively prime to the polynomial $x + 1$.

Polynomials that are *not* relatively prime to $x + 1$ must be multiples of $x + 1$, since $x + 1$ is irreducible. A polynomial in $GF(2)[x]$ is a multiple of $x + 1$ if and only if its value at $x = 1$ is 0. But $r(1) = 0$ if and only if $r(x)$ has an even number of non-zero coefficients. Therefore $r(x)$ is relatively prime to $f_n(x)$ if and only if k is odd.

Thus, when n is a power of two, R is an invertible operation on $GF(2)^n$ if and only if k is odd. \blacksquare

2 Discussion

The inverse operation to R can be found using Euclid's extended algorithm on input polynomials $r(x)$ and $f_n(x)$, to find polynomials $s(x)$ and $t(x)$ such that

$$s(x) \cdot r(x) + t(x) \cdot f_n(x) = 1 .$$

The inverse operation S to R corresponds to the polynomial $s(x)$, representing another function of the same form as R (that is, an xor of rotations). In matrix terms, the inverse of a circulant matrix is another circulant matrix.

In terms of computational complexity, $R(v)$ is easy to compute when k is small, requiring not more than k rotations and $k - 1$ xors. Although the inverse S has the same form as R , it may require considerably more work to compute. For example, if $r(x)$ has degree d , then $s(x)$ must have degree at least n/d and at least n/d terms, so that evaluating $S(v)$ requires at least $\log_2(n/d)$ additions, since each addition in a computation chain can at most double the number of terms. Here multiplication by x^r (rotations) are “free” and we are only counting exclusive-ors. The exact complexity, in terms of rotations and xors, of evaluating $R(v)$ or $S(v)$ may be non-trivial to determine precisely, and we leave these questions as open problems. Thus, when k and d are small R may be considered to be in some sense “very modestly one-way”—easier to compute in one direction than another. Stephen Boyack [3] has interesting related results on the complexity of matrix operations over $GF(2)$ and their inverses.

Efficient invertible operations are useful in many applications. A linear operation somewhat similar to the one studied here is the “xorshift” operation:

$$v = v \oplus (v \ll r)$$

where “ \ll ” is the “left-shift” operator; xorshift has been used in pseudo-random number generation [7, 9] and hash-function design [10]. Schnorr and Vaudenay [11, Lemma 5] study the related operation

$$(v \wedge d) \oplus (v \lll r)$$

where “ \wedge ” denote bitwise “and” and where d is a constant n -bit word; they show that this operation is invertible if and only if the iterates $(d \lll (r \cdot i))$ take for each bit position the value 0 for some i .

The result of this paper may be useful to those working on similar applications. For example, we began our study of R when thinking about possible improvements to the MD6 hash function [10]. We also note that the $k = 3$ version of the operation discussed here is used in the C2 cipher [2] (although not in manner that required its invertibility (it is part of the feedback function in a Feistel block-cipher)), and in the SHA hash function standard message expansion computation [8] (as the Σ function; invertibility of Σ is not claimed or proven).

When n is not a power of 2, we don’t know of any comparably simple characterization of when $R(v)$ is invertible, other than the requirement that $\gcd(f_n(x), r(x)) = 1$; perhaps simpler characterizations can be found for some cases, such as when $n = 3 \cdot 2^k$.

3 Related Work

Lars Knudsen points out that a different proof for the same result is available in the the Ph.D. thesis [12, Theorem 3.3, pages 86–87] of Søren Thomsen. Thomsen’s cute proof considers powers R^{2^i} of the original operation, notes that

$$R^2(v; r_1, r_2, \dots, r_k) = R(v; 2r_1, 2r_2, \dots, 2r_k)$$

from which it follows that R is invertible since R^n will be the identity function (if and only if k is odd).

4 Conclusions

This note provides an alternate proof of a characterization as to when an easily computed operation, based on the exclusive-or of rotated versions of a word, is invertible.

Acknowledgments

I’d like to thank Lars Knudsen, Niels Ferguson, Louay M. J. Bazzi, and Sanjoy K. Mitter for helpful comments and suggestions.

References

- [1] Dario Bini, Gianna M. Del Corso, Giovanni Manzini, and Luciano Margara. Inversion of circulant matrices over Z_m . *Mathematics of Computation*, 70(235):1169–1182, Mar 24 2000.
- [2] Julia Borghoff, Lars R. Knudsen, Gregor Leander, and Krystian Matusiewicz. Cryptanalysis of C2. In S. Halevi, editor, *Proc. CRYPTO’09*, volume 5671 of *Lecture Notes in Computer Science*, pages 250–266. Springer, 2009.

- [3] Stephen Wayne Boyack. *The Robustness of Combinatorial Measures of Boolean Matrix Complexity*. PhD thesis, MIT Mathematics Dept., 1985.
- [4] Scott Contini, Ronald L. Rivest, M.J.B. Robshaw, and Yiqun Lisa Yin. Improved analysis of some simplified variants of RC6 . In Knudsen, editor, *Proc. Fast Software Encryption '99*, volume 1636 of *Lecture Notes in Computer Science*, pages 1–15. Springer, 1999.
- [5] Pu hua Guan and Yu He. Exact results for deterministic cellular automata with additive rules. *J. Stat. Physics*, 14(3/4):463–478, 1986.
- [6] Rudolf Lidl and Harald Niederreiter. *Finite Fields*. Cambridge University Press, 1983. Vol. 20 of Encyclopedia of Mathematics and its Applications.
- [7] G. Marsaglia. Xorshift RNGs. *J. Stat. Soft.*, 8(14):1–6, 2003. <http://www.jstatsoft.org/v08/i14/xorshift.pdf>.
- [8] National Institute of Standards and Technology. Secure hash standard, August 1, 2002. FIPS 180-2.
- [9] François Panneton and Pierre L'Ecuyer. On the Xorshift random number generators. *ACM Trans. Modeling and Computer Simulation*, 15(4):346–361, Oct. 2005.
- [10] Ronald L. Rivest, Benjamin Agre, Daniel V. Bailey, Christopher Crutchfield, Yevgeniy Dodis, Kermin Elliott Fleming, Asif Khan, Jayant Krishnamurthy, Yuncheng Lin, Leo Reyzin, Emily Shen, Jim Sukha, Drew Sutherland, Eran Tromer, and Yiqun Lisa Yin. The MD6 hash function: A proposal to NIST for SHA-3, Sep 2008. <http://groups.csail.mit.edu/cis/md6/>.
- [11] Claus P. Schnorr and Serge Vaudenay. Black box cryptanalysis of hash networks based on multipermutations. In *Proceedings EUROCRYPT '94*, volume 950 of *Lecture Notes in Computer Science*, pages 47–57. Springer, 1995.
- [12] Søren Steffen Thomsen. *Cryptographic Hash Functions*. PhD thesis, Technical University of Denmark, November 28, 2008.