ASYMPTOTIC BOUNDS FOR THE NUMBER OF CONVEX $n$-OMINOES

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Abstract. Unit squares having their vertices at integer points in the Cartesian plane are called cells. A point set equal to a union of $n$ distinct cells which is connected and has no finite cut set is called an $n$-omino. Two $n$-ominoes are considered the same if one is mapped onto the other by some translation of the plane. An $n$-omino is convex if all cells in a row or column form a connected strip. Letting $c(n)$ denote the number of different convex $n$-ominoes, we show that the sequence \((c(n))^{1/n} : n = 1, 2, \ldots\) tends to a limit $\gamma$, and $\gamma = 2.309138\ldots$.

1. Introduction

Unit squares having their vertices at integer points in the Cartesian plane are called cells. A point set equal to a union of $n$ distinct cells which is connected and has no finite cut set is called an $n$-omino. Two $n$-ominoes are considered the same if one is mapped onto the other by some translation of the plane. (Such $n$-ominoes were called fixed animals with $n$ cells by Read [8].) For example, there are six different 3-ominoes as shown in Fig. 1.

![Fig. 1. The 3-ominoes.](image)

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Let \( t(n) \) denote the number of distinct \( n \)-ominoes. It is known [3] that the sequence \( (t(n))^{1/n} : n = 1, 2, \ldots \) tends to a limit \( \theta \). The investigation of \( \theta \) began with Eden’s work [1]; he managed to prove that \( 3.14 < \theta \leq 6.75 \). There has been considerable effort expended to improve these bounds. Currently, the best lower bound (given in [3]) is \( 3.72 < \theta \), while the best upper bound (given in [5]) is \( \theta < 4.65 \).

An \( n \)-omino is row-convex when each row of the \( n \)-omino is a connected strip of cells. Column-convex \( n \)-ominoes are defined analogously. All six of the 3-ominoes (shown in Fig. 1) are both row-convex and column-convex; in general, such \( n \)-ominoes are said to be row-column-convex, or just convex for short. It was shown in [2] (and in [3] by a second method) that

\[
\frac{x(1-x)^3}{1-4x+7x^2-5x^3} = \sum_{n=1}^{\infty} b(n) x^n,
\]

where \( b(n) \) denotes the number of distinct row-convex \( n \)-ominoes. (This result was also obtained by Pólya [6].) Thus it follows that the sequence \( (b(n))^{1/n} : n = 1, 2, \ldots \) tends to a limit \( \beta \) which is equal to the largest real root of

\[
y^3 - 4y^2 + 7y - 5 = 0;
\]

that is, \( \beta = 3.20\ldots \).

Recently, Donald Knuth wrote us and asked us if the number \( c(n) \) of convex \( n \)-ominoes had been investigated. This paper is entirely motivated by Knuth’s question. We shall be concerned with the problem of effectively calculating the limit \( \gamma \) of the sequence \( (c(n))^{1/n} : n = 1, 2, \ldots \) . One of the first things we prove is that this limit exists. Later on we show how to calculate upper and lower bounds for \( \gamma \) and give the best results obtained by these methods.

2. Existence of \( \lim_{n \to \infty} (c(n))^{1/n} \)

Following Caesar’s admonition, we divide, then conquer. A convex \( n \)-omino may be split into three parts by making two cuts between certain rows so that the upper and lower parts are roughly trapezoids and the middle part is roughly a parallelogram. A typical sectioning of this sort is shown in Fig. 2. More precisely, the trisection of a convex \( n \)-omino \( A \) is accomplished by cutting along the lowest level of \( A \) where the left boundary of \( A \) goes to the right and by cutting along the lowest level of \( A \) where the right boundary of \( A \) goes to the left.
A convex \( n \)-omino whose left boundary climbs to the right and whose right boundary climbs to the left corresponds to a partition of \( n \) called a \textit{stack} by Wright [9]. We let \( s(n) \) denote the number of distinct \( n \)-ominoes corresponding to stacks; for example, there are four 3-ominoes shown in Fig. 1 which correspond to stacks, so \( s(3) = 4 \). A convex \( n \)-omino whose left and right boundaries both climb to the right is called a \textit{parallelogram}, and \( p(n) \) will denote the number of distinct \( n \)-ominoes which are parallelograms. Clearly, \( p(n) \leq c(n) \) for all \( n \); also, \( s(n) \leq p(n) \) for all \( n \) (the diagram in Fig. 3 suggests a proof of this fact). Finally, an obvious construction establishes that \( p(m) \, p(n) \leq p(m+n) \) for all \( m, n \).

Now we use the fact that if \( \{u_n\} \) is a sequence of natural numbers such that \( (u_n)^{1/n} : n = 1, 2, \ldots \) is bounded and \( u_m \, u_n \leq u_{m+n} \) for all \( m, n \), then \( \lim_{n \to \infty} (u_n)^{1/n} \) exists. (For similar results, see Pólya and Szegő...
[7, p. 171].) We have \( p(n) \leq b(n) < (3.20)^n \) for all large \( n \), and 
\( p(m) p(n) \leq p(m+n) \), so

\[
(2) \quad \lim_{n \to \infty} (p(n))^{1/n} = \gamma
\]

exists. Using the fact that every convex \( n \)-omino splits into two stacks and one parallelogram, we can reconstruct these \( n \)-ominoes by pasting together two stacks and one parallelogram in various ways. Again, using an obvious construction, and using the fact that \( p(i) p(j) p(k) \leq p(i+j+k) \) for all \( i, j, k \), it is easy to show that

\[
(3) \quad c(n) \leq 2n^2 \sum_{(i,j,k)} s(i) p(j) s(k) \leq 2n^2 \sum_{(i,j,k)} p(i) p(j) p(k)
\]

\[
\leq 2n^2 (n+2)^2 p(n) \leq (n+2)^4 p(n),
\]

where the index of summation in the sums extends over all compositions \( (i, j, k) \) of \( n \) into non-negative parts. There are \( \binom{n+2}{2} \) such compositions.

Using (2) and (3) together with the fact that \( p(n) \leq c(n) \) for all \( n \), we have

\[
(4) \quad \gamma = \lim_{n \to \infty} (p(n))^{1/n} \leq \lim_{n \to \infty} \inf (c(n))^{1/n} \leq \lim_{n \to \infty} \sup (c(n))^{1/n} \leq \lim_{n \to \infty} ((n+2)^4 p(n))^{1/n} = \gamma.
\]

Hence \( \lim_{n \to \infty} (c(n))^{1/n} \) exists, and

\[
(5) \quad \lim_{n \to \infty} (c(n))^{1/n} = \lim_{n \to \infty} (p(n))^{1/n} = \gamma.
\]

3. An integral equation

We shall use a theory developed in [4] concerning a double sequence 
\((b(n, a) : n, a = 1, 2, \ldots)\) defined in terms of given sequences 
\((f(m, n) : m, n = 1, 2, \ldots)\) and \((g(n) : n = 1, 2, \ldots)\) as follows:

\[
(6) \quad b(n, a) = \sum f(a_1, a_2) f(a_2, a_3) \ldots f(a_{k-1}, a_k) g(a_k),
\]
where the index of summation extends over all \( k \)-tuples \((a_1, \ldots, a_k)\) of natural numbers for \( k = 1, \ldots, n \) with \( a_1 = a \) and \( a_1 + \ldots + a_k = n \). It was shown that if

\[
G(x) = \sum_{n=1}^{\infty} g(n) x^n
\]

and

\[
F(x, y) = \sum_{m, n=1}^{\infty} f(m, n) x^m y^n
\]

converge for \(|x| \) and \(|y| \) sufficiently small, then

\[
B(x, y) = \sum_{n=1}^{\infty} \sum_{a=1}^{n} b(n, a) y^a x^n
\]

converges for \(|x| \) and \(|y| \) sufficiently small, and

\[
B(x, y) = G(xy) + \frac{1}{2\pi i} \int_{C} F(xy, 1/s) B(x, s) \frac{ds}{s},
\]

where \( C \) is a contour in the \( s \)-plane which includes \( s = 0 \) and the singularities of \( F(xy, 1/s) \) but excludes the singularities of \( B(x, s) \). The theory of (10) runs parallel to that of the Fredholm integral equation. In particular, if \( F(x, y) \) has the special form

\[
F(x, y) = R_1(x) S_1(y) + \ldots + R_r(x) S_r(y),
\]

we say \( F \) is \textit{separable}, and it turns out that (10) can be converted into a system of \( t \) equations linear in \( t \) unknown functions. The system can be solved and the solution yields a formula for \( B(x, y) \). We shall give an example of this later on.

If \( F \) is not separable, we can still get information about \( B \) by approximating \( F \) with something that is separable. Suppose

\[
K(x, y) = \sum k(m, n) x^m y^n
\]

and \( k(m, n) \leq f(m, n) \) for all \( m, n \), then we say \( K \) is a \textit{lower bound} on \( F \), an \textit{upper bound} on \( F \) is defined analogously. If \( K \) is separable, we may
substitute $K$ for $F$ in (10) and calculate a lower bound for $B$. Upper bounds for $B$ may be obtained in a similar fashion. We shall adopt this strategy too, so an example is forthcoming.

The relevance of the foregoing discussion to the enumeration of $n$-celled parallelograms is as follows: the number of $(m+n)$-celled parallelograms having $m$ cells in one row and $n$ cells in a second row is

$$f(m,n) = \min \{m, n\}. \tag{13}$$

It is fairly easy to show that the number of $n$-celled parallelograms with exactly $k$ rows of cells having exactly $a_i$ cells in the $i$th row for $i = 1, \ldots, k$ is

$$f(a_1, a_2) f(a_3, a_3) \ldots f(a_{k-1}, a_k). \tag{14}$$

Thus, if we take $f$ as defined in (13) and put $g(j) = 1$ for all $j$, we can sum (6) over $a = 1, \ldots, n$ and obtain $p(n)$. In this case, we have

$$F(x, y) = xy/(1-x)(1-y)(1-xy), \tag{15}$$

$$G(x) = x/(1-x). \tag{16}$$

Substituting these functions in (10) gives

$$B(x, y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_C \frac{xy B(x, s) ds}{(1-xy)(s-1)(s-xy)} \tag{17}$$

$$= \frac{xy}{1-xy} + \frac{xy}{(1-xy)^2} B(x, 1) - \frac{xy}{(1-xy)^2} B(x, xy).$$

We can iterate (17) to eliminate $B(x, xy), B(x, x^2y), \ldots$ successively to find

$$B(x, y) = \sum_{k=1}^\infty (-1)^{k+1} \frac{x^k(k+1)/2 y^k(1-x^k y + B(x, 1))}{(1-xy)^2 (1-x^2y)^2 \ldots (1-x^k y)^2}. \tag{18}$$

Setting $y = 1$ in (18), we solve for $B(x, 1)$, the generating function of $(p(n): n = 1, 2, \ldots)$, which turns out to be
\[ B(x, 1) = \frac{x}{1-x} \frac{x^3}{(1-x)^2} \frac{x^6}{(1-x)^2 (1-x^2)^2} \frac{x^6}{(1-x)^2 (1-x^2)^2 (1-x^3)^2} + \ldots \]

\[ = \sum_{n=1}^{\infty} p(n) x^n . \]

We have been unable to make use of (19) in estimating \( p(n) \). Instead, we use upper and lower bounds for \( F \) as defined in (15), and then use (10) to calculate upper and lower bounds for \( B \).

4. Lower bounds

Let

\[ F_k(x, y) = \sum_{m,n=1}^{k} f(m, n) x^m y^n , \]

where \( f(m, n) = \min \{ m, n \} \) just as in (13), and let \( B_k(x, y) \) denote the solution of (10) having \( F_k \) substituted for \( F \). Since \( F_k \) is a lower bound for \( F \), it follows that \( B_k \) is a lower bound for \( B \). It was shown in [4] that when the kernel of (10) is approximated by a polynomial as in this case, then \( B_k(x, 1) \) is a rational function, say \( B_k = P_k/Q_k \) with \( P_k \) and \( Q_k \) polynomials, and the denominator of \( B_k \) may be expressed as a determinant. In the present situation this turns out to be

\[ Q_k(x) = \begin{vmatrix} 1-x & 1 & 1 & \ldots & 1 \\ 1 & 2-x^2 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & \ldots & \ldots & 1 \end{vmatrix} \]

If we put \( Q_0(x) = 1 \) and \( Q_1(x) = 1-x \), we can use (21) to verify that

\[ Q_k(x) = (1-x^{k-1} - x^k) Q_{k-1}(x) - x^{2k-2} Q_{k-2}(x) \]
for \( k = 2, 3, \ldots \). For example,

\[
Q_2(x) = 1 - 2x - x^2 + x^3,
\]
\[
Q_3(x) = 1 - 2x - 2x^2 + 2x^3 + 2x^4 + x^5 - x^6,
\]
\[
Q_4(x) = 1 - 2x - 2x^2 + x^3 + 3x^4 + 5x^5 - 2x^6 - 2x^7 - 2x^8 - x^9 + x^{10}.
\]

Letting \( \gamma_k \) denote the largest real root of \( Q_k(1/x) = 0 \), we have \( \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma \), where \( \gamma \) is defined in (2). We have used a computer to calculate lower bounds for \( \gamma_1, \gamma_2, \ldots, \gamma_{10} \) given in Table 1. Our results indicate that the sequence \( \{\gamma_l\} \) converges very quickly to the value 2.30913859..., our best lower bound for \( \gamma \).

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5. Upper bounds

For \( k = 1, 2, \ldots \), we define upper bounds \( f^k(m, n) \) for \( f(m, n) = \min \{m, n\} \) as follows:

\[
(25) \quad f^k(m, n) = \begin{cases} 
  m & \text{if } k < n < m, \\
  f(m, n) & \text{otherwise.}
\end{cases}
\]

Hence

\[
(24) \quad F^k(m, n) = \sum_{m,n=1}^{\infty} f^k(m, n) x^m \ y^n = \frac{xy}{(1-x)^2(1-y)} - \frac{x^2y}{(1-x)^2} - \ldots - \frac{x^{k+1} \ y^k}{(1-x)^2}
\]
is an upper bound for $F$; furthermore, note that $F^k$ is separable.

Let $B^k$ denote the solution of (10) with $F^k$ substituted for $F$. Then

$$B^k(x, y) = \frac{xy}{1 - xy} + \frac{xyB^k(x, 1)}{(1 - xy)^2} = \frac{xy}{(1 - xy)^2} \sum_{r=1}^{k} x^r y^r B^k_r(x),$$

where

$$B^k_r(x) = \frac{1}{k!} \left. \frac{\partial^r}{\partial s^r} B^k(x, s) \right|_{s=0}.$$  

Now we use (25) to get a system of equations involving $B^k_1, \ldots, B^k_k$. Take the $r$th partial derivative with respect to $y$ at $y = 0$ and divide by $r!$ in (25) to get

$$B^k_r(x) = x^r + r x^r B^k(x, 1) - \sum_{j=1}^{r-1} (r-j) x^r B^k_j(x),$$

from which it follows that

$$B^k_{r+1}(x) = (2x - x^{r+1}) B^k_r(x) - x^2 B^k_{r-1}(x).$$

Setting $B^k_r(x) = P_r(x) + Q_r(x) B^k(x, 1)$ for $r = 1, \ldots, k$, it follows that $P_r$ and $Q_r$ also satisfy the difference equation (27). Also we can substitute $P_r + Q_r B^k$. For $B_r$ in (25) with $y = 1$ and solve for $B^k(x, 1)$ in terms of $P_1, Q_1, \ldots, P_k, Q_k$ to obtain

$$B^k(x, 1) = \frac{x - x^2 - \sum_{j=1}^{k} x^{j+1} P_j(x)}{1 - 3x + x^2 + \sum_{j=1}^{k} x^{j+1} Q_j(x)}.$$  

Thus $B^k$ is a rational function whose numerator $N_k$ and denominator $D_k$ we know how to compute because they are defined in terms of $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ which we know how to compute. Let $\beta_k$ denote the largest real root of $D_k(1/x)$, then we know

$$\lim_{n \to \infty} \left( \sum_{a=1}^{n} b^k(n, a) \right)^{1/n} = \beta_k \leq \gamma,$$

and $\beta_1 \geq \beta_2 \geq \ldots \geq \gamma$. Thus we can calculate upper bounds for $\beta_1, \beta_2, \ldots$ to obtain successively better upper bounds for $\gamma$. 

Using the definitions

\begin{align}
D_k &= 1 - 3x + x^2 + x^2 Q_1 + \ldots + x^{k+1} Q_k, \\
Q_{r+1} &= (2x - x^{r+1}) Q_r - x^2 Q_{r-1} \quad (r > 1),
\end{align}

and $Q_1 = x$, $Q_2 = 2x^2 - x^3$, the polynomials $D_1, D_2, \ldots$ are calculated with relative ease. For example, we found

\begin{align*}
D_1 &= 1 - 3x + x^2 + x^3, \\
D_2 &= 1 - 3x + x^2 + x^3 + 2x^5 - x^6, \\
D_3 &= 1 - 3x + x^2 + x^3 + 2x^5 - x^6 + 3x^7 - 2x^8 - 2x^9 + x^{10}.
\end{align*}

Using a computer, the polynomials $D_1, \ldots, D_{10}$ were calculated via (30), and upper bounds for $\beta_k$, the largest real root of $D_k(1/x) = 0$, were computed for $1 \leq k \leq 10$ using the Newton–Raphson method. These upper bounds for $\beta_k$ are given in Table 1.

Combining our upper and lower bounds, we can conclude that

\begin{equation}
\gamma = \lim_{n \to \infty} \frac{c(n)}{n}^{1/n} = 2.309138\ldots.
\end{equation}

References