18.211

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These are my lecture notes from 18.211, Combinatorial Analysis, at the Massachusetts Institute of Technology, taught this semester (Fall 2018) by Professor Yufei Zhao[1].

I wrote these lecture notes in LaTeX in real time during lectures, so there may be errors, typos or omissions. I have lovingly pillaged Evan Chen's[2] and Tony Zhang's[3] formatting commands. Should you encounter an error in the notes, wish to suggest improvements, or alert me to a failure on my part to keep the web notes updated, please contact me at rmwu@mit.edu.

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1 September 13, 2018

I added this class late, so welcome to lecture 3!

> **Question 1.1.** A comment about problem set 1—what is an ordered pair?

An ordered pair (x, y) is a pair of non-distinct numbers x, y that is ordered.

1.1 Partitions

The general setup is simple: we have n balls and k bins, and we would like to count the number of ways to put balls in bins. Of course, there are some details to hammer out: are the balls distinct, and are all the bins distinct? There are four combinations and we will consider all four of them.

1.1.1 Compositions

Suppose we have identical balls and distinct bins. More generally, suppose we have a sum (order matters) that adds up to a total number.

Definition 1.2. A **weak composition** of n is a sequence $(a_1, a_2, \ldots, a_k)$ of non-negative integers that add up to some number n. If all the a_i are positive, then this sequence is known as a **composition**.

Example 1.3

What is the number of weak compositions of n into k parts?

We can use the "stars and bars" techniques.

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

This is equal the number of ways to arrange n into k TODO $\binom{n+k-1}{n}$

Example 1.4

What is the number of compositions of n into k parts?

Now we *could* use this technique directly, but we need constraints that disallow adjacent bars and bars at the start and end. Instead, we can use a bijection: first put one ball into every bin, then we have $n - k$ balls left. Thus, we have $\binom{n-1}{n-k} = \binom{n-1}{k-1}$.

Example 1.5

What is the total number of compositions of n?

We could sum up all the possibilities:

$$\sum_{k\geq 1}\binom{n-1}{k-1} = 2^{n-1}.$$

Alternately, we could have n stars with $n-1$ positions for bars, so there are 2^{n-1} choices.

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$$

We are constructing an implicit bijection between stars-and-bars diagrams and the number of compositions.

Definition 1.6. Now suppose we have n distinct balls and k identical boxes. A **set partition** partitions $[n] = \{1, 2, \ldots, n\}$ into k nonempty subsets.

(1) (2) (3) (4) $\cdots$ (6)

Example 1.7

Suppose $n = 4, k = 2$.

We can count by hand first. There are 7 of these.

$$\begin{array}{ll}
\{1,2,3\},\{4\} & \text{choose 1 loner} \\
\{1,2,4\},\{3\} & \\
\{1,3,4\},\{2\} & \\
\{2,3,4\},\{1\} & \\
\{1,2\},\{3,4\} & \text{pairs} \\
\{1,3\},\{2,4\} & \\
\{1,4\},\{2,3\} &
\end{array}$$

Definition 1.8. The **Stirling number of the second kind** $S(n, k)$ is the number of partitions of $[n]$ into k nonempty subsets.

We counted manually that $S(4, 2) = 7$. What about the general case? Let's do some obvious examples first.

$$\begin{aligned}
S(n,k) &= 0 \text{ if } n < k \\
S(0,0) &= 1 \\
S(n,1) &= 1 \\
S(n,n) &= 1 \\
S(n,n-1) &= \binom{n}{2}
\end{aligned}$$

Sadly, there are no closed formula for the general $S(n, k)$. However, we can still find useful relationships.

Theorem 1.9

$S(n,k) = S(n-1,k-1) + kS(n-1,k)$

Proof. We count the number of partitions of $[n]$ into k nonempty sets. Consider the element n. Either n is in a set by itself, or it has friends. If n is a loner, then we have $S(n-1,k-1)$ ways to partition the rest. Otherwise, consider all $S(n-1,k)$ partitions of the remaining elements. There are k different places to place n, so we have $k \cdot S(n-1,k)$. □

Example 1.10

What is the number of surjective functions from $[n] \to [k]$?

Back to the balls and bins example, we have distinct balls and distinct bins, with nonempty bins (surjective). If the bins were identical, then we simply have $S(n,k)$ partitions. Now there are $k!$ ways to label the bins, so we have $k!S(n,k)$.

Corollary 1.11

For all $x \in \mathbb{R}$,

$$x^n = \sum_{k=0}^{n} S(n,k) \cdot x(x-1)(x-2)\dots(x-k+1).$$

Proof. Both sides are polynomials of degree at most n, so it suffices to show that they agree at $n+1$ different values of x. In fact, we check this for all $x \in \mathbb{N}$.

Consider the number of ways to partition n distinct balls into x distinct bins. Let k be the number of nonempty bins. There are $\binom{x}{k}$ ways to select these bins. There are $S(n,k) \cdot k!$ ways to put balls into these k bins. We sum over $k = 0, 1, \dots, n$. □

Definition 1.12. The n^{th} **Bell number** $B(n)$ is the number of partitions of $[n]$ into nonempty sets.

In terms of Stirling numbers,

$$B(n) = \sum_{k=1}^{n} S(n,k).$$

By convention, $B(0) = 1$.

Theorem 1.13

Suppose we know the first n Bell numbers. Then

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i).$$

Proof. $B(n+1)$ represents the number of partitions of $[n+1]$. Now consider element $n+1$. How large is the size of the block (partition) that contains $n+1$?

- If $n+1$ is a loner, there are $B(n)$ ways to partition the rest.
- If $n+1$ has a single partner, there are $n \cdot B(n-1)$ to partition the rest.

In general, if $n+1$ lies in a block of size $i+1$, there are $\binom{n}{i}$ ways to select the partners and $B(n-i)$ ways to partition the rest. Thus, if we sum all these possibilities, we obtain the desired result. □

1.1.2 Integer partitions

Definition 1.14. A partition of the number n is a sequence $a_1, a_2, \ldots, a_k$ of positive integers with $a_1 \geq a_2 \geq \cdots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = n$.

It is important to note that set partitions of $[n]$ are *different* from integer partitions of n.

Integer partitions are equivalent to n identical balls and n identical bins. Since these are identical bins, we canonically sort the bins into order of descending number of balls. For example,

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 2+1+1 \\ &= 2+2 \\ &= 3+1 \\ &= 4. \end{aligned}$$

Let $p(n)$ be the number of partitions of n, and $p_k(n)$ be the number of partitions of n into exactly k parts. There is no closed form formula, but Ramanujan found that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\tau\sqrt{n/6}}.$$

We often visualize partitions by Ferrers shapes or Young diagrams. For example, the partition $(4,2,1)$ may be visualized as

The conjugate of a partition can be determined as follows: we draw the main diagonal and reflect across that line. The conjugate of $(4,2,1)$ is $(3,2,1,1)$.

Some partitions are self-conjugate—that is, it is its own conjugate. For example, $(5,2,1,1,1)$ is self-conjugate.

Theorem 1.15

The number of partitions of n into at most k parts is equal to the number of partitions of n into parts each not larger than k.

Proof. Consider the Young diagrams. We are simply looking at the bijection between Young diagrams and their conjugates. The left hand side is the number of diagrams with at most k rows, and the right hand side is the number of diagrams with at most k columns. □

This is known as a bijection via conjugation.

Theorem 1.16

The number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions.

Proof. Let's start with an example.

$$\begin{aligned} 15 &= 15 \\ &= 11 + 3 + 1 \\ &= 9 + 5 + 1 \\ &= 7 + 5 + 3 \end{aligned}$$

Let's draw the pictures! We take the rows, bend them into Ls, and stack them. We can also unbend the Ls and unstack them. □

To summarize, today we talked about balls and bins. We have n balls and k bins, which may be distinct or identical.

	n balls distinct	identical
dictinct	surjections, $S(n,k) \cdot k!$	compositions, $\binom{n-1}{k-1}$, 2^{n-1}
identical	set partitions, $S(n,k)$, $B(n)$	integer partition, $p_k(n)$, $p(n)$

2 September 18, 2018

Let's start with a puzzle.

Example 2.1 (100 prisoners)

We have 100 prisoners and an evil warden. There is a room with 100 closed boxes, each with a different name. The prisoners are asked, one at a time, to go into the room and open 50 boxes, one at a time. After the prisoner leaves the room, there is no more communication. If every prisoner finds his or her own name, then everyone is free to go. Otherwise, everyone dies. What is the optimal strategy?

There is a naive strategy: everyone picks 50 boxes at random, but this has a very low rate of success, $1/2^{100}$. Luckily, there exists a strategy with success probability $> 30\%$. We will talk about this today, but first, let's learn some math.

2.1 Permutations

We will focus on the cycle structure of permutations. Previously, we've viewed permutations as lists or orderings. However, we can also view permutations as bijective functions from $[n] \to [n]$.

Example 2.2

Consider $123 \mapsto 312$. "312" is known as one-line notation, or "word form." However, we can also view this map as a function $f : [3] \to [3]$ where

$$f(1) = 3 \quad f(2) = 1 \quad f(3) = 2.$$

The one-line notation has a cousin known as the two-line notation,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The functional form has some nice properties, notably, composition.

Example

Let $f = 312, g = 213$. Then $fg = 132$ and $gf = 321$.

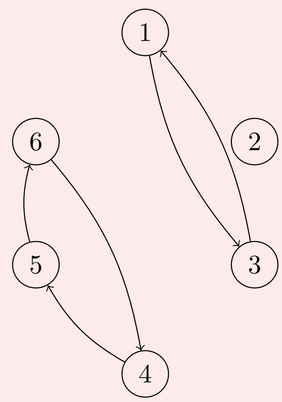
Note that this convention is a different convention from the textbook, but it is standard in algebra. In this class, we are only interested in counting permutations; we care less about the group's structure.

2.1.1 Cycle notation

In addition to this functional notation, we can express notations as cycles.

Example

Let $g = 321564$.



We may say that 2 is a fixed point, $\{1, 3\}$ form a 2-cycle, and $\{4, 5, 6\}$ form a 3-cycle. Thus, we denote g as $(13)(456)$.

The notation $g = (13)(456)$ is known as product notation. These cycles are disjoint, so they can be composed in any order.

Remark 2.3. We do not include parentheses in word-form, but we do include parentheses for cycle notation. This is important for disambiguation.

Every permutation can be decomposed into disjoint cycles. This is proved in the textbook but we will not discuss this here.

Example

Consider the permutation $(134)(25)(67)(8)$. It's two-line notation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 1 & 2 & 7 & 6 & 8 \end{pmatrix}$$

and the one-line notation is the second line.

2.1.2 100 prisoners riddle

Now we can solve the 100-prisoners puzzle with observations about cycles. Consider the following strategy.

1. Prisoners randomly assign themselves numbers $1, 2, \ldots, 100$ and assign the boxes numbers.
2. Each prisoner opens his "own" box. If he finds his own slip, he is done. Otherwise, he opens the box corresponding to the name inside.
3. Repeat until he finds his name, or exceeds 50 boxes.

What is the probability of success? They fail if there is a cycle that exceeds 50.

Let $r > 50$. How many permutations of $[100]$ have a length of exactly r? There can only be one cycle of length r since $100 - r < 50$. There are $\binom{100}{r}$ ways to select r numbers, and $(r-1)!$ cycles (there are $r!$ permutations but cyclic rotations). There are $(100-r)!$ permutations for the rest.

So we have $\frac{100!}{r}$ permutations with a cycle of length r. The success rate is

$$1-\sum_{r=51}^{100}\frac{1}{r}\approx 0.311.$$

Now consider this variation. Suppose we have $2n$ prisoners and we can open n boxes. Then our rate of success is

$$1-\sum_{r=n+1}^{2n}\frac{1}{r}\to 1-\log 2\approx 0.306.$$

This is a beautiful answer, so our combinatorial professor would like a beautiful story. What are some questions we can ask to get that story?

Suppose π is a random permutation of $[n]$.

1. What is the probability that 1 and 2 are in the same cycle?
2. What about 1,2, and 3?
3. What is the probability that 1 is in a cycle of length k?

2.1.3 Cycle type

The symmetric group S_n peeks out its ugly head again. For this class, we can consider it the set of permutations of $[n]$.

Definition 2.4. For $\pi \in S_n$, if π has exactly a_i cycles of length $i, \forall i = 1, 2, \ldots, n$, then π has **cycle type** $(a_1, a_2, \ldots, a_n)$.

Example 2.5

Consider the permutation $(412)(53)(76) \in S_8$. This has type

$$(1,2,1,0,0,0,0,0).$$

We make a few observations.

- If $\pi \in S_n$ has type $(a_1, a_2, \ldots, a_n)$, then $\sum_{i=1}^{n} a_i i = n$, where we sum up the number of elements over cycles.
- The number of cycles is $\sum_i a_i$.
- The number of fixed points is a_1.

Theorem 2.6

Let $a_1, \ldots, a_n$ be a set of non-negative integers with $\sum_{i=1}^{n} i a_i = n$. The number of permutations in S_n with this cycle type is

$$\frac{n!}{\prod_{i=1}^{n} i^{a_i} a_i!}.$$

Proof. Suppose we write down a permutation of $[n]$ into empty slots for each of the cycles. Each cycle of length i can be rotated i ways. There are $a_i!$ ways to rearrange cycles of length i. □

Example 2.7

The number of permutations with exactly one cycle is $(n-1)!$.

This is of type $(0, \ldots, 0, 1)$, so we can plug it straight into the theorem. We can also fix the first element, then find $(n-1)!$ ways to rearrange the rest.

2.1.4 Stirling numbers of the first kind

Definition 2.8. The number of permutations $[n]$ with exactly k cycles is known as the **signless Stirling number of the first kind**, denoted as $c(n, k)$.

Definition 2.9. A **Stirling number of the first kind** is denoted as $s(n, k) = (-1)^{n-k} c(n, k)$.

Example 2.10

Let's look at few easy examples.

- By definition or convention or truth, $c(0, 0) = 1$.
- $c(n, 0) = 0$ if $n > 0$.
- $c(n, k) = 0$ if $k > n$.
- $c(n, n) = 1$, the identity.
- $c(n, n-1) = \binom{n}{2}$.

Recall from last time that we defined Stirling numbers of the second kind $S(n, k)$ as the number of partitions of $[n]$ into k blocks. We found that $S(n, k) = S(n-1, k-1) + kS(n-1, k), \forall 0 < k \leq n$.

Likewise, we have a recursion for Stirling numbers of the first kind.

Theorem 2.11

$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), \forall 0 < k \leq n$.

Proof. Consider the element n. There are two cases.

1. n is a fixed point. The remaining $[n-1]$ can be permuted into $c(n-1, k-1)$.
2. n lies in a cycle of length at least 2. We write down the cycle decomposition and erase n. Now we have a permutation of $[n-1]$ with k cycles. To restore n, we specify the number preceding n, of which there are $n-1$.

Combining the two cases, there are $c(n-1, k-1) + (n-1)c(n-1, k)$ ways to partition $[n]$ into k cycles. □

Stirling numbers of the first and second kinds are actually related in an intrinsic, mathematical way!

Theorem 2.12

For $n \in \mathbb{N}, x \in \mathbb{R}$,

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\dots(x+n-1).$$

We will prove this next lecture, but let's play with this proposition a bit. If we replace x with $-x$, we get

$$\sum_{k=0}^{n} c(n,k)(-1)^k x^k = (-x)(-x+1)(-x+2)\dots(-x+n-1).$$

If we multiply by $(-1)^n$, then we have

$$\sum_{k=0}^{n} s(n,k)x^k = x(x-1)(x-2)\dots(x-n+1).$$

Last time, we learned that

$$\sum_{k=0}^{n} S(n,k)x(x-1)(x-2)\dots(x-n+1) = x^n.$$

There is strong sense in which $S(n,k)$ and $s(n,k)$ are inverses of each other. The standard basis for polynomials we see is $1, x, x^2, \dots, x^n$. However, there are others, such as the falling factorial basis $1, x, x(x-1), x(x-1)(x-2), \dots$.

The Stirling numbers provide the linear transformations between these two bases, and in fact, they are inverses of each other.

$$\begin{pmatrix} s(0,0) & s(0,1) & s(0,2) & \dots \\ s(1,0) & s(1,1) & s(1,2) & \dots \\ \vdots & & & \end{pmatrix}$$

3 September 20, 2018

3.1 Permutations, ctd.

Let's review the facts from last lecture.

- We learned that permutations can also be represented as cycles and functions.
- $c(n, k)$ is the number of permutations of $[n]$ with exactly k cycles, and this is known as the signless Stirling number of the first kind.
- Theorem 2.11 stated that

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), \forall 0 < k \leq n.$$

Today we will learn a few tricks to help us solve problems about permutations. For example, what fraction of the permutations of $[n]$ have 1 and 2 in the same cycle? The professor proclaims that he'll show us how to do it, "in a snap."

Recall theorem 2.12 from the last lecture, repeated below.

Theorem

For $n \in \mathbb{N}, x \in \mathbb{R}$,

$$\sum_{k=0}^{n} c(n, k)x^k = x(x+1)(x+2)\dots(x+n-1).$$

Proof. We will show that the coefficient of x^k on the right hand side satisfies the same recurrence relation as $c(n, k)$.

Let $G_n(x) = x(x+1)(x+1)\dots(x+n-1) = \sum_{k=0}^{n} a_{n,k}x^k$ where $G_0(x) = 1$. We want to show that $a_{n,k}$ satisfies the same relation as $c(n, k)$. The polynomial G_n can also be defined recursively as $G_n(x) = (x+n-1)G_{n-1}(x)$. So

$$\sum_{k=0}^{n} a_{n,k}x^k = (x+n-1)\sum_{k=0}^{n-1} a_{n-1,k}x^k.$$

We can expand the polynomials and compare the coefficients of x^k:

$$a_{n,k} = a_{n-1,k-1} + (n-1)a_{n-1,k}.$$

Lucky for us, this is the same recurrence! How we check the base cases. By definition, $a_{0,0} = 1$, and $a_{n,0} = 0$ since there are no constant terms. Therefore, $a_{n,k} = c(n, k)$ agree for all relevant values. □

While this proof is correct, it doesn't provide as much intuition. So our professor will give us a combinatorial proof, which is quite clever and he doesn't expect us to come up with it on our own.

Proof. The Chinese restaurant process describes the following:

> You're at a Chinese restaurant with k tables and you join a table. Whenever a new person comes in, they can either join an existing table, or they can start a new table. If they join an additional table, he has a choice of seat (on a round table).

After n customers arrive, we have a cycle decomposition corresponding to a permutation of $[n]$. Recall that k is the number of tables, or the number of cycles. Now let us extend the problem.

> Suppose that every table has a menu of x items, and each table selects exactly 1 item.

So we arrive at the standard question: how many ways are there to do this?

First consider the number of ways to sit. If there are k tables, then there are $c(n,k)$ ways to sit, and x^k ways to choose the dishes. Thus, there are

$$\sum_{k=0}^{n} c(n,k)x^k.$$

Now let us consider each customer as they come in.

1. The first person comes in, sits, and orders food.
2. The second person has two choices. Either he joins the first person and eats the same food, or he starts his own table and orders new food.
3. The third person can join, or he can start a new table and order new food.

□

3.1.1 Canonical cycle form

Now let's talk about canonical cycle form. Consider the cycle (125)(876)(49). We could as correctly write (512) instead of (125), or switched the orders. What could we do to make this format more standard?

Definition 3.1. The **canonical cycle form** gives us a convention for writing cycle decompositions. Included all fixed points. For each cycle, write the largest element first. Then sort the cycles in increasing order of their first element.

For our previous example, we would have (512)(876)(94). Now we can count cycles more easily!

Definition 3.2. For permutation $\pi \in S_n$, let $\hat{\pi} \in S_n$ be the permutation obtained by writing π in the canonical form and reading it in one-line form.

Example 3.3

Let $\pi = 43268175$. In cycle notation, we have

$$\begin{aligned} 43268175 &= (146)(23)(58)(7) \\ &= (614)(32)(85)(7) \\ &= (32)(614)(7)(85) \text{ canonical form} \\ \hat{\pi} &= 32614785. \end{aligned}$$

In fact, this transformation is a bijection from S_n to itself. We hand wave and show that its inverse exists.

Proof. Find the start of the cycles by taking the left-to-right maxima. For example, we have (32)(614)(7)(85). If π has k cycles, then $\hat{\pi}$ has k left-to-right maxima.

As a corollary, the distribution of the number of cycles is the same as the distribution of left-to-right maxima, over all $n!$ permutations. □

Recall the question we posed at the beginning of the lecture.

Example 3.4

What fraction of the permutations of $[n]$ have 1 and 2 in the same cycle?

Let us relabel $1 \to n, 2 \to n-1$. So our question becomes, what fraction of permutations of n have n and $n-1$ in the same cycle?

In the canonical cycle form, n is always a left-to-right maximum in $\hat{\pi}$. and in particular, n is the start of the right-most cycle. $n-1$ lies in the same cycle of n if and only if $n-1$ lies to the right of n.

Now we may ask, what fraction of permutations has n before $n-1$ in one-line form? Of course, the answer is $1/2$.

Example 3.5

Let k, n be positive integers where $k \leq n$. What fraction of permutations of $[n]$ has the element 1 in a k cycle?

Relabel $1 \to n$ and count the number of permutations where n is the k^{th} element from the end. So the answer is $1/n$.

Example 3.6

For $r > n/2$, what is the probability that a random permutation of $[n]$ has an r cycle?

The expected number of elements in an r cycle is $1/n \cdot n = 1$. There are r elements in the r cycle, so

$$r \cdot \Pr\{\exists r \text{ cycle}\} = 1$$

and our probability is $1/r$.

Let $\mathsf{Odd}(n)$ be the set of permutations of $[n]$ such that all cycle lengths are odd, and let $\mathsf{Even}(n)$ be the set of permutations of $[n]$ such that all cycle lengths are even. In particular, the latter has no fixed points.

Theorem 3.7

$|\mathsf{Odd}(2m)| = |\mathsf{Even}(2m)| = 1^2 3^2 \ldots (2m-1)^2$.

Proof. We give a bijection from Odd to Even.

Starting with $\pi \in \mathsf{Odd}$, π must have an even number $2k$ odd cycles, $c_1, c_2, \ldots, c_{2k}$. We take the last element of c_i and append it to c_{i+1}, for $1 \le i \le k$. It is evident that this new $\pi' \in \mathsf{Even}$.

To go in reverse, we read π' from the right. For cycle c_i, if its last element is smaller than the first element of c_{i-1}, then it came from c_{i-1}. Else, it came from its own singleton cycle.

Now we start from $\pi \in \mathsf{Even}$. Suppose we start from element 1. There are $2m-1$ choices to go out from 1, and $2m-1$ choices to go back. From here, if we went back, there are $2m-3$ choices for the new cycle, if we went back, or $2m-3$ choices for the next element in the same cycle. We see that at each step i, the number of total choices is $2m-(2i-1)$. ☐

Remark 3.8. This bijection was reasonable to construct, and it was easy to count even cycles. It is a lot harder to count odd cycles.

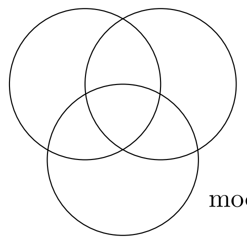
4 September 25, 2018

Let's start with a quick example.

Example 4.1

How many elements of [100] are divisible by 2, 3, or 5?

"You probably don't need to take this course to figure this out, but you can draw a Venn diagram."



Let's count the elements divisible by each.

2	50
3	33
5	20
2 and 3	16
2 and 5	10
3 and 5	6
2, 3, 3	3

Putting it all together, we find that our answer is

$$50 + 33 + 20 - 16 - 10 - 6 + 3 = 74.$$

4.1 Sieve

This method is known as a **sieve**, which comes from sifting for prime numbers.[4]

Theorem 4.2 (The principle of inclusion-exclusion)

Let $A_1, \dots, A_n$ be finite sets. Then

$$\begin{aligned} |A_1 \cup \dots \cup A_n| &= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \dots \\ &\quad + (-1)^{n-1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{j-1}^{n} (-1)^{j+1} \sum_{1 \le i_1 < \dots < i_j \le n} \left| A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j} \right| \end{aligned}$$

[4] The Sieve of Eratosthenes. https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes

Proof. We show that every element gets counted the same number of times on the two sides of the equation.

Elements outside this union are never counted, so we may ignore them. Let $x \in A_1 \cup \cdots \cup A_n$. The left hand side obviously counts x once.

Now consider the right hand side. Let S be the set of A_i that contain x and let $s = |S|$. We see that

$$s = \binom{s}{2} + \binom{s}{3} + \cdots + (-1)^{s+1}\binom{s}{s}.$$

Recall from the second lecture that

$$1 - \binom{s}{2} + \binom{s}{3} - \cdots + (-1)^{s+1}\binom{s}{s} = 0,$$

so $s = 1$.

□

Now let's work on some examples.

4.1.1 Derangements

Each of n guests at a party has a hat, and when leaving, everyone takes an arbitrary hat. No one leaves with their own hat. In how many ways can this happen?

Definition 4.3. A **derangement** of $[n]$ is a permutation of $[n]$ without fixed points. Let $D(n)$ be the number of derangements of $[n]$.

There are some easy cases.

- $D(1) = 0$, this is trivial.
- $D(2) = 1$, they swap hats.
- $D(3) = 2$, there are two 3-cycles.
- $D(4) = 8$, there are 6 4-cycles and 3 ways to swap (involution).

It's kind of hard to count permutations without fixed points, so instead we count permutations with fixed points.

Let A_i be the set of permutations π of $[n]$ with $\pi(i) = i$. Then we know that

$$D(n) = n! - |A_i \cup A_2 \cup \cdots \cup A_n|.$$

Let's look at some concrete cases.

- $|A_1| = (n-1)!$, since we can assign the rest arbitrarily.
- $|A_1 \cap A_2| = (n-2)!$, since we fix two and assign the rest.
- For j different indices, we have $(n-j)!$ ways to assign the rest.

We have that

$$\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= n(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! + \cdots + (-1)^{n-1} \\
&= n! - \frac{n!}{2!} + \frac{n!}{3!} + \cdots + (-1)^{n-1} \\
&= n!\left(1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right).
\end{aligned}$$

Therefore, the number of derangements is

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{n!}\right).$$

This is $D(n)$, but the professor has a little more to say. Recall that the Taylor expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

and for $1/e$,

$$\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = \sum_k \frac{(-1)^k}{k!}.$$

So we find that as $n \to \infty$, the probability that a random permutation is a derangement is $1/e$. It also turns out that $D(n)$ is the closest integer of $n!/e$.

4.1.2 Stirling numbers of the second kind, revisited

Recall that $S(n,k)$ (Stirling numbers of the second kind) is the number of partitions of $[n]$ into k blocks.

Theorem 4.4

The Stirling number of the second kind can be defined as

$$\begin{aligned}
S(n,k) &= k^n - \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}(k-i)^n \\
&= \frac{k^n}{k!} - \sum_{i=1}^{k}\frac{(-1)^{i+1}(k-i)^n}{i!(k-i)!}
\end{aligned}$$

Also recall that the number of surjections from $[n] \to [k]$ is $k! \cdot S(n,k)$, since we make a partition and may arbitrarily permute them. A surjective function never "misses" an element in $[n]$, so let A_i be the set of functions $[n] \to [k]$ that "miss" i in the image. Then the number of surjections is

$$k^n - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

This is a natural setup for inclusion-exclusion! For all sequences $1 \le i_1 < i_2 < \cdots < i_j \le n$,

$$\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}\right| = ???$$

$$= \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n + \cdots + (-1)^k\binom{k}{k}(k-k)^n.$$

4.1.3 Euler totient function

The Euler totient function $\phi(n)$ is the number of elements of $[n]$ that are relatively prime to n. We start with some examples to find a formula for ϕ.

- $\phi(3) = 2$, since we exclude 3.
- $\phi(6) = 2$, since we exclude 2,3,4, and 6.
- Suppose p is prime. Then $\phi(p) = p - 1$.
- Suppose p, q are distinct primes. Then
$$\phi(pq) = pq - q - p + 1.$$
- Suppose p, q, r are distinct primes. Then
$$\phi(pqr) = pqr - pq - qr - pr + p + q + r - 1.$$
- $\phi(p^2) = p^2 - p = p(p-1)$.
- $\phi(p^2q) = p^q - pq - p^2 + p = p(p-1)(q-1)$.

Now we can work on a general formula. Let $n = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ be the prime factorization of n, where $p_1, \dots, p_k$ are different primes and $m_1, \dots, m_k$ are positive integers. Note that "relatively prime to n" implies "not divisible by $p_1, p_2, \dots, p_k$".

Let A_i be the set of integers in $[n]$ that are divisible by p_i.

- $|A_i| = n/p_i$.
- $|A_{i_1} \cap A_{i_2}| = \frac{n}{p_{i_1} p_{i_2}}$ counts the elements divisible by $p_{i_1} p_{i_2}$.
- If we take a j-fold intersection, we simply have n divided by the corresponding p_i.

In general,

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{1 \le j \le k} \frac{n}{p_{i_j}} - \sum_{1 \le j_1 < j_2 \le n} \frac{n}{p_{j_1} p_{j_2}} + \dots$$

However, we care about the complement of this set, as

$$\begin{aligned} \phi(n) &= n - |A_1 \cup \cdots \cup A_n| \\ &= n(1 - \text{the long sum from above}) \\ &= \end{aligned}$$

5 September 27, 2018

5.1 Generating functions

Generating functions are a powerful tool in enumerative combinatorics, and they often represent the answer to many problems.

Let's begin with some easy examples.

Example 5.1

Let $a_{n+1} = 2a_n + 1, \forall n \geq 0, a_0 = 0$.

Each time we multiply by 2 and add one, giving us

n	0	1	2	3	...	n
a_n	0	1	3	7	...	$2^n - 1$

where we can probe the last term by... guess and check. That's not too great a strategy, so let's learn about generating functions instead.

Definition 5.2. Given sequence $a_0, a_1, a_2, \dots$, the associated generating function is

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

With this definition, let us revisit the example. We write out the recurrence and convert it into a generating function.

$$\begin{aligned} a_{n+1} &= 2a_n + 1 \\ a_{n+1}x^{n+1} &= 2a_n x^{n+1} + x^{n+1} \\ \sum_{n \geq 0} a_{n+1}x^{n+1} &= \underbrace{\sum_{n \geq 0} 2a_n x^{n+1}}_{2xA(x)} + \underbrace{\sum_{n \geq 0} x^{n+1}}_{x/(1-x)}. \end{aligned}$$

We solve and find that

$$A(x) = \frac{x}{(1-x)(1-2x)}.$$

This doesn't give us the answer just yet, but we can use partial fractions to derive the exact formula for a_n:

$$\frac{x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}.$$

If we multiply both sides by $1-x$, we find that $A = -1$, and if we multiply by $1-2x$, then we find that $B = 1$. Now we can expand the terms as geometric series,

$$\begin{aligned} A(x) &= \frac{1}{1-2x} - \frac{1}{1-x} \\ &= \left(1 + 2x + 2^2x^2 + \dots\right) + \left(1 + x + x^2 + \dots\right) \\ &= \sum_{n \geq 0} (2^n - 1)x^n. \end{aligned}$$

Since the coefficients must match, $a_n = 2^n - 1$.

Remark 5.3. "Depending on your level of mathematical maturity...

If you're just starting out, you might think of these as functions, and you can take derivatives. Physicists are very comfortable with this. If you're further along—if you've taken 18.100B—you might be worried, can we take derivatives? But if you're even further along, you'll know that the power series form a ring and things actually work out! Some of you may be stuck in that awkward stage where you feel uncomfortable, so just trust me, it works."

Generally, we do not evaluate generating functions. They are formal power series. For example, the series $1/(1-x)$ does not evaluate at $x = -1$, but in terms of generating functions, we think of it as

> "the function that, multiplied with $1 - x$, gives you 1."

5.1.1 Strategy for solving recurrences

We provide a general strategy for working with generating functions.

Example 5.4

Let $a_{n+2} = 3a_{n+1} - 2a_n, \forall n \geq 0, a_0 = 0, a_1 = 1$.

1. Write down the generating function in terms of the series.

$$A(x) = \sum_{n\geq 0} a_n x^n$$

2. Manipulate the recurrence, e.g. multiply by x^n or similar, then sum over all valid n.

$$\begin{aligned} a_{n+2}x^{n+2} &= 3a_{n+1}x^{n+2} - 2a_n x^{n+2} \\ \sum_{n\geq 0} a_{n+2}x^{n+2} &= \sum_{n\geq 0} 3a_{n+1}x^{n+1} - \sum_{n\geq 0} 2a_n x^{n+2} \\ \sum_{n\geq 2} a_n x^n &= 3x \sum_{n\geq 1} a_n x^n - 2x^2 \sum_{n\geq 0} a_n x^n. \end{aligned}$$

3. Rewrite in terms of $A(x)$.

$$A(x) - a_0 - a_1 x = 3x(A(x) - a_0) - 2x^2 A(x)$$

   Do not forget the initial conditions.

4. Expand the generating function $A(x)$.

$$A(x) = \sum_{n\geq 0} (2^n - 1)x^n$$

5. Find the closed form solution. This is the same function as last time, so we conclude that $a_n = 2^n - 1$.

Example 5.5

Let $a_{n+1} = 2a_n + n, \forall n \geq 0, a_0 = 1$.

Let's write down a few terms.

$$\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & \dots \\ a_n & 1 & 2 & 5 & 12 & 27 & \end{array}$$

We follow the steps again.

1. $A(x) = \sum_{n\geq 0} a_n x^n$. It's good to remind ourselves what the variable is.
2. Now we manipulate the function,

$$\begin{aligned} a_{n+1}x^{n+1} &= 2a_n x^{n+1} + nx^{n+1} \\ \underbrace{\sum_{n\geq 0} a_{n+1}x^{n+1}}_{A(x)-a_0} &= \underbrace{\sum_{n\geq 0} 2a_n x^{n+1}}_{2xA(x)} + \underbrace{\sum_{n\geq 0} nx^{n+1}}_{x^2/(1-x)^2}. \end{aligned}$$

   The last term is a bit nontrivial to determine. However, we know that $1/(1-x) = \sum_{n\geq 0} x^n$, so we can take its derivative to obtain that

$$\frac{\partial}{\partial x}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{n\geq 1} nx^{n-1}.$$

3. So we can write

$$\begin{aligned} A(x) - 1 &= 2xA(x) + \frac{x^2}{(1-x)^2} \\ A(x) &= \frac{(1-x)^2 + x^2}{(1-2x)(1-x)^2}. \end{aligned}$$

4. We want to decompose as

$$\frac{1-2x+2x^2}{(1-2x)(1-x)^2} = \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x}.$$

   We multiply through by $(1-x)^2$ and plug in $x = 1$ to obtain $A = -1$. Then we multiply through by $1-2x$ and plug in $x = 1/2$ to obtain $C = 2$. Now there's a third term left, and you *cannot* plug in $x = 1$, else the A term tends to ∞. We resort to standard algebra, plugging in A and C, and we find that $B = 0$. So

$$A(x) = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}.$$

5. Let's expand the terms. We get

$$1/(1-x)^2 = \sum_{n\geq 1} nx^{n-1} = \sum_{n\geq 0}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

   by differentiating $1/(1-x)$. We also get that

$$2/(1-2x) = \sum_{n\geq 0} 2^{n+1}x^n$$

from our previous problem. So combining,

$$A(x) = \sum_{n\geq 0} \left(2^{n+1} - n - 1\right) x^n$$

Finally, we find that $a_n = 2^{n+1} - n - 1$.

5.2 Product formula for generating functions

So far we've been playing with generating functions algebraically, but they have meaningful combinatorial implications as well.

We can multiply generating functions. Let

$$A(x) = \sum_{n\geq 0} a_n x^n$$
$$B(x) = \sum_{n\geq 0} b_n x^n.$$

The product AB is a generating function whose coefficient of x^n is

$$(ab)_n = \sum_{i=0}^{n} a_i b_{n-1} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

What does this mean to us?

Example 5.6

A semester has n days. How many ways can we split a semester into two contiguous parts, where the first part consists of the first k days and the second part contains the remaining $n-k$ days. Additionally, we choose a holiday from the first part and two holidays from the second part.

Let f_n be the answer. Naively, as we usually solve these,

$$f_n = \sum_{n=1}^{n-2} k \binom{n-k}{2}.$$

If we try hard enough, we can simplify this, but let's use some generating functions. For the first part, let

$$A(x) = \sum_{k\geq 0} k x^k.$$

We read this off as "if the first part has k days, there are k choices." Likewise,

$$B(x) = \sum_{m\geq 0} \binom{m}{2} x^m$$

which is read off as "if the second part has m days, we choose two of them." We claim that our answer is just the product,

$$\sum_{n\geq 0} f_n x^n = A(x)B(x).$$

If we expand the generating function, the coefficient of x^n comes from k in A and $n-k$ in B.

We know that

$$A(x) = \frac{x}{(1-x)^2}$$
$$\frac{1}{(1-x)} = \sum_{n\geq 0} x^n$$
$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

Now let's look at $B(x) = x^2/(1-x)^3$. The product is

$$\sum_{n\geq 0} f_n x^n = A(x)B(X) = \frac{x^3}{(1-x)^5}$$

This tells us that $f_n = \binom{n+1}{4}$.

Theorem 5.7 (Product formula)
Suppose a_n is the number of ways to have a certain structure on an n-element set, and b_n is the number of ways to have another such structure. Let c_n be the number of ways to split $[n]$ into two intervals (may be empty), where we build the first type of structure on the first half, and same for the second half. If $A(x), B(x), C(x)$ are the corresponding generating functions, then $A(x)B(x) = C(x)$.

In the previous example, $a_n = n, b_n = \binom{n}{2}$.

Why is this true? Well

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} x^k$$

which is true both in the combinatorial interpretation and by the generating function definition.

Example 5.8
Let A choose any number of holidays in the first part, B choose an odd number of holidays, and C choose an even number of holidays.

The generating functions are

$$A(x) = \sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}$$
$$B(x) = \sum_{n\geq 1} 2^{n-1} x^n = \frac{x}{1-2x}$$
$$C(x) = \sum_{n\geq 0} 2^{n-1} x^n = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}.$$

We can expand partial fractions, which the professor spared us from.

$$\frac{x(1-x)}{(1-2x)^3} = \sum_{n\geq 0} \binom{n+2}{2} x^n = \underbrace{\sum_{n\geq 0} \frac{1}{4}\left(-2^n + \binom{n+2}{2} 2^n\right)}_{\text{this is what we want}} x^n.$$

Since there's 5 more minutes, let's give a preview for next time. Let $p(n)$ be the number of partitions of n. It turns out that

$$\sum_{n\geq 0} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}$$

which is very beautiful!

6 October 2, 2018

We have an in-class midterm next week on Thursday, since Tuesday is a holiday for Columbus Day. It will be closed book, with 6 problems over 80 minutes. A practice exam will be posted later today.

Last time, we learned that the generating function associated with a sequence $(a_n)_{n\geq 0}$ is $A(x) = \sum_{n\geq 0} a_n x^n$, which is a formal power series (we do not evaluate it). Today we will focus on combinatorial interpretations of generating functions.

6.1 Generating functions for partitions

Recall that $p(n)$ is the number of partitions of n, where each partition is a set of integers that add to n.

Suppose we define j_i as the number of times i occurs in a partition of n. Consider the generating function

$$(1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots$$

where we have the powers of x, x^2, x^3, etc. Imagine expanding the power of x^n as

$$x^n = x^{1j_1+2j_2+3j_3+\cdots+kj_k}.$$

The exponent is a partition of n. We will see that the quantity above is the generating function for the number of partitions. More succinctly, we may say

$$\sum_{n\geq 0} p(n)x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots = \prod_{n\geq 0} \frac{1}{1-x^n}. \tag{6.1}$$

What does this mean, though, and how can we use it? Well suppose we only care about the number of partitions of n into parts of sizes at most b. Then we truncate the equation above to

$$\sum_{n=0}^{k} p(n)x^n = \prod_{n=0}^{b} \frac{1}{1-x^n}. \tag{6.2}$$

Claim 6.1. Equation 6.2 also represents the generating function for partitions of n with at most b parts.

Looking at the generating function, this is not immediately obvious. Instead, recall our discussion on Young diagrams from lecture 1 (theorem 1.15). We can draw a bijection between Young diagrams and their complements, and this claim becomes immediately apparent.

Theorem 6.2

The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Proof. We show that the generating functions for both sides are the same. The generating function A for the left hand side is the same as before, with even factors skipped:

$$A(x) = (1 + x + x^2 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^5 + x^{10} + \dots)\dots \tag{6.3}$$

The generating function B for the right hand side corresponds to a generating function

$$B(x) = (1+x)(1+x^2)(1+x^3)\dots \tag{6.4}$$

in which no factor is repeated—that is, the js are either 0 or 1, so we are left with terms like $(x^2)^0, (x^2)^1$ and no powers of $(x^2)^3$, etc.

We can rewrite A with our favorite algebraic identity for $1/(1-x)$ and rewrite B as a product of difference of squares,

$$\begin{aligned}
A(x) &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \dots \\
B(x) &= \frac{(1-x)(1+x)}{1-x} \cdot \frac{(1-x^2)(1+x^2)}{1-x^2} \cdot \frac{(1-x^3)(1+x^3)}{1-x^3} \dots \\
&= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \dots \\
&= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \dots \\
&= A(x).
\end{aligned}$$

Thus, we see that A and B are equal, and there are equal partitions of n into odd parts as partitions of n into distinct parts.

□

While this algebraic proof is straightforward, it is not as illuminating as a combinatorial proof, so the professor offers us a bijection as well.

Proof. Given a partition into distinct parts, we factor each part as an odd part, multiplied by a power of 2. For example,

$$30 = 12 + 7 + 6 + 4 + 1 = 3 \cdot 2^2 + 7 + 3 \cdot 2 + 1 \cdot 2^2 + 1.$$

Then split each odd part with multiplicity, the factor of two.

$$30 = (3+3+3+3) + 7 + (3+3) + (1+1+1+1) + 1.$$

We now have our partition into odd parts.

To prove that this mapping is a bijection, we show that this function has an inverse. Given a partition into odd parts, we group factors by multiplicity,

$$30 = 7 \cdot 1 + 3 \cdot 6 + 1 \cdot 5.$$

We write the multiplicities in base 2,

$$30 = 7 \cdot 1 + 3 \cdot (2^2 + 2) + 1 \cdot (2^2 + 1).$$

Expand as

$$30 = 7 + 3 \cdot 2^2 + 3 \cdot 2 + 1 \cdot 2^2 + 1 = 7 + 12 + 6 + 4 + 1$$

and we restore our original partition. Magic!

We always get distinct parts because we have distinct odd parts, and each binary expansion is unique. Thus, we have proven our desired statement via bijection.

□

6.2 Catalan numbers

We cannot take a class on combinatorics without visiting the Catalan numbers at least once, so here we are. First, a motivating example.

Example 6.3

Imagine that we have two candidates in an election, $\{0, 1\}$. There are n votes for 0 and n votes for 1. In how many different orders can the voters cast their votes, so that 0 never trails 1?

For $n = 3$, we could have 000111 but not 101010. In fact, we can write out all the valid orders for $n = 3$.

000111	001101	010011
010101	001011	

We can also write these as parenthesizations, where we never close more parentheses than we open. For example, 001011 is (()()). Now let's count.

Suppose that the first "unit" has i open (and i closed). For example,

$$\underbrace{(()()(()))}_{\text{unit with } i=5}\ ()(()).$$

Note that the first unit is still a valid expression after removing the first (and last). There are c_{i-1} ways to choose the first unit (since we only choose the parentheses inside the outermost pair) and c_{n-i} ways to choose the remaining. By summing over all i, we find that

$$c_n = \sum_{i=1}^{n} c_{i-1}c_{n-i}$$

with the convention that $c_0 = 1$. The associated generating function is

$$C(x) = \sum_{n\geq 0} c_n x^n.$$

We write the recurrence

$$\underbrace{\sum_{n\geq 1} c_n x^n}_{C(x)-1} = \underbrace{\sum_{n\geq 1}\sum_{i=1}^{n} c_{i-1}c_{n-i}x^n}_{xC(x)C(x)}.$$

We claim that $C(x) = 1 + xC(x)C(x)$ since

$$xC(x)C(x) = x\sum_{k\geq 0} c_k x^k \cdot \sum_{j\geq 0} c_j x^j = \sum_{k\geq 1} c_{k-1}x^k \cdot \sum_{j\geq 0} c_j x^j$$

This is quadratic in C, so we solve

$$xC^2 - C + 1 = 0$$

$$C = \frac{1 \pm \sqrt{1-4x}}{2x}. \tag{6.5}$$

One root is correct, but how do we know which one? Well we can expand both, and the constant term had better be c_0.

$$\frac{1+\sqrt{1-4x}}{2x} \quad \text{vs.} \quad \frac{1-\sqrt{1-4x}}{2x}$$

Sometimes we can plug in $x = 0$ for the constant term, but here that's nonsense (division by 0). The right hand side is

$$\frac{1 - (1 - 4x/2 + O(x^2) + \dots)}{2x} = 1 + O(x^n).$$

That's okay. The left hand side is not okay because we get some factor of $1/x$, so $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. Well that's nice, but we want a formula for the Catalan numbers. Let's expand $(1-4x)^{1/2}$ by the binomial theorem.

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-n+1\right)}{n!} = \frac{(-1)^{n-1}1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n n!}.$$

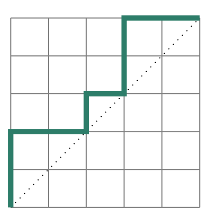
Finally, we obtain that

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \forall n \geq 0. \tag{6.6}$$

Looking at this formula, we're not even sure this an integer! But our combinatorial explanation tells us that this number counts something.

It turns out that our now-retired Richard Stanley wrote a whole book about Catalan numbers, which contains 214 combinatorial interpretations of Catalan numbers. Here are a few examples, for culture.

1. Catalan numbers also count the number of Dyck paths: we start at (0,0) and at each step, either move up or right, until we end up at (n, n).

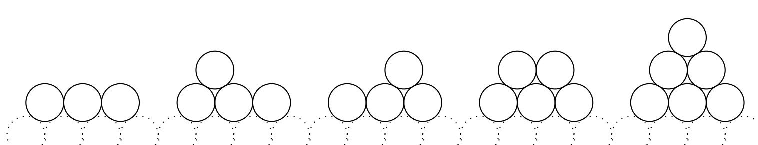
   

   Alternatively, we can start at (0,0) and move one step right, and one step up or down each time, until we end up at $(2n, 0)$. To prove this bijection, we can map an open parenthesis (to up and closed parenthesis) to down.

   ()()() (())() ()(()) (()()) ((()))

2. We could count valid can stackings—this was a homework problem.

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6.2.1 Compositions

Now that we have the basics down, we can use generating functions to count cool problems. We'll start with generating functions of compositions. Let

$$F(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots$$

be a function we will become intimately familiar with. Given some $G(x)$ with 0 constant term,

$$F(G(x)) = \frac{1}{1-G(x)} = 1 + G(x) + G(x)^2 + \dots$$

In particular, we generally prefer all our generating functions without constant terms. This format is quite useful, as we will see.

Example 6.4

Suppose we have n soldiers in a line, and some officer cuts the line into several non-empty groups and selects one leader from each group. How many different ways can we do this?

Let $G(x) = \sum_{k\geq 1} nx^n$ be the generating function for the number of ways to select an officer in a group of n soldiers. Consider the function,

$$F(G(x)) = 1 + G(x) + G(x)^2 + G(x)^3 + \dots = \frac{1}{1-G(x)}. \tag{6.7}$$

How can we interpret this combinatorially?

- The constant factor 1 represents the number of ways we can solve this problem with no groups (this is convention).
- $G(x)$ is the generating function that counts this problem, if we have one group (i.e. we make no cuts).
- $G(x)^2$ counts two groups, ...and so on.

The exponents of $F(G(x))$ cover all possible values of n. To obtain the number of ways for any n, we simply take the coefficient for x^n.

Now we would like to find a closed form solution for g_n. The closed form solution of $G(x)$ is $x/(1-x)^2$, as we have found before, so

$$1 + \frac{x}{1-3x+x^3}. \tag{6.8}$$

We may complete the square to solve that

$$g_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \tag{6.9}$$

where $\alpha = (3+\sqrt{5})/2, \beta = (3-\sqrt{5})/2$.

7 October 4, 2018

7.1 Generating functions, ctd.

Today we will focus on compositions of generating functions. Last time we saw an example with soldiers in a line. This time, we will use that as an example to describe the general case.

Theorem 7.1

Let us consider the **compositional formula** for generating functions.

- Let a_n be the number of ways to build a certain structure on $[n]$.
- Let b_n be the number of ways to build a second structure.
- Let g_n be the number of ways to split $[n]$ into non-empty intervals. We build a structure of the first type on each of these intervals, and we build a structure of the second type on the set of intervals.
- Let A, B, G be the corresponding generating functions.
- By convention, we set $a_0 = 0, b_0 = 1, g_0 = 1$.

Then

$$G(x) = B(A(x)).$$

Example (6.4)

Suppose we have n soldiers in a line, and some office cuts the line into several non-empty groups and selects one leader from each group. How many different ways can we do this?

Last time, $a_n = n$ represented the number of ways to select a leader, and $b_n = 1$ represented $B(x) = 1/(1-x)$, so $G(x) = 1/(1 - A(x))$.

Proof. We may expand

$$B(A(x)) = b_0 + \underbrace{b_1 A(x)}_{\text{1 interval}} + \underbrace{b_2 A(x)^2}_{\text{2 intervals}} + \dots$$

We claim that this is equal to $\sum_{n\geq 0} g_n x^n$. If we expand each term of $A(x)$, blah blah oops I spaced out lmao. □

Example 7.2

Suppose we have n soldiers in a line, and some office cuts the line into several non-empty groups. Then some non-empty subset of the groups are chosen for night duty. How many different ways can we do this?

In this case, $a_n = 1$ could represent "do nothing" with an interval, and $b_n = 2^n - 1$ could represent the subsets chosen. With care taken for initial conditions,

$$A(x) = x + x^2 + x^3 \cdots = \frac{x}{1-x}$$
$$B(x) = 1 + x + 3x^2 + \cdots = \frac{1}{1-2x} - \frac{x}{1-x}.$$

So composing,

$$G(x) = B(A(x)) = \frac{1}{1 - \frac{2x}{1-x}} - \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}}.$$

Notice that this method is very versatile!

- If instead, we allowed an empty subset for night duty, then we wouldn't have the right term that comes from $x/(1-x)$.
- We could also choose an officer for each group, in addition to night duty. Then $a_n = n$.
- Suppose we select an officer *and* order each group. Then $a_n = n!$.

While these problems are cool, it seems more natural to work with unordered sets. To do this, we need to introduce some notation.

7.2 Exponential generating functions

Definition 7.3. Given sequence $(a_n)_{n\geq 0}$, the associated **exponential generating function** is

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

Why do we call this exponential? If $a_n = 1$, then its associated function is

$$\sum_{n\geq 0} \frac{x^n}{n!} = e^x.$$

Example 7.4

Let $a_0 = 1, a_{n+1} = (n+1)(a_n - n + 1), \forall n \geq 0.$

We define $A(x) = \sum_{n\geq 0} \frac{a_n x^n}{n!}$. We write the recurrence and sum over n,

$$\sum_{n\geq 0} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n\geq 0} (n+1)(a_n - n + 1) \frac{x^{n+1}}{(n+1)!}.$$

Let's understand these terms a bit more.

$$\underbrace{\sum_{n\geq 0} a_{n+1} \frac{x^{n+1}}{(n+1)!}}_{A(x)-a_0} = \underbrace{\sum_{n\geq 0} (n+1)(a_n}_{a_n x^{n+1}/n!} - \underbrace{\sum_{n\geq 0} (n+1)(n-1) \frac{x^{n+1}}{(n+1)!}}_{xxxx}$$
$$= xxx$$

We solve for A to obtain

$$\begin{aligned} A(x) &= \frac{1}{1-x} x e^x \\ &= \sum_{n\geq 0} x^n + \sum_{n\geq 0} \frac{x^{n+1}}{n!} \\ &= \sum_{n\geq 0} \end{aligned}$$

The coefficients we want are those of $x^n/n!$, which are

$$a_n = n! + n.$$

Example 7.5

Let $f_0 = 0, f_{n+1} = 2(n+1)f_n + (n+1)!, \forall n \geq 0.$

We define $F(x) = \sum_{n\geq 0} \frac{f_n x^n}{n!}$. We write the recurrence and sum over n. These terms come out very cleanly!

$$\underbrace{\sum_{n\geq 0} f_{n+1} \frac{x^{n+1}}{(x+1)!}}_{F(x)-f_0} = \underbrace{\sum_{n\geq 0} 2f_n \frac{x^{n+1}}{(x+1)!}}_{2xF(x)} + \underbrace{\sum_{n\geq 0} x^{n+1}}_{\frac{x}{1-x}}$$

After partial fraction decomposition,

$$\begin{aligned} F(x) &= \frac{x}{(1-x)(1-2x)} \\ &= \frac{-1}{1-x} + \frac{1}{1-2x} = \sum_{n\geq 0} (2^n - 1)x^n. \end{aligned}$$

So our term is $f_n = (2^n - 1)n!$.

7.2.1 Product of exponential generating functions

Products of ordinary exponential functions were related to structures, so what do products mean for exponential generating functions? Let's look at the algebra first.

Lemma 7.6

If A, B, C are the exponential generating functions of sequences a_n, b_n, c_n, respectively, and $C = AB$, then

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_n b_{n-i}, \forall n \geq 0.$$

Proof. The exponential generating function is the ordinary generating function of a modified sequence, $a_n/n!$, etc. We may apply the product formula for ordinary generating functions to obtain our result. □

The factor of $\binom{n}{i}$ is very nice! It lets us give combinatorial explanations.

Example 7.7

Suppose we have n people, split into two groups. We ask each group to form a line, and then we ask each member of the first group to choose a color (red/green/blue) as their personal shirt color. How many ways can we do this?

This is not a hard problem, and we could just find the answer directly.

Suppose our two groups have sizes $k, n-k$. There are $\binom{n}{k}$ ways to pick the split, $k!$ ways to order the first group, and 3^k ways to select colors. There are $(n-k)!$ ways to order the second group. That gives us

$$\sum_{k=0}^{n} \binom{n}{k} k!(n-k)!3^k.$$

Now that we know the answer, let's use generating functions.

$$A(x) = \sum_{k\geq 0} k!3^k \cdot \frac{x^k}{k!} = \frac{1}{1-3x}$$

$$B(x) = \sum_{j\geq 0} j! \cdot \frac{x^j}{j!} = \frac{1}{1-x}$$

We claim that AB is our answer. It follows directly from our formula. With partial fractions, we find that

$$A(x)B(x) = \frac{1}{(1-x)(1-3x)} = \sum_{n\geq 0} \left(\underbrace{\frac{3^{n+1}-1}{2} n!}_{\text{our answer}} \cdot \frac{x^n}{n!} \right).$$

Theorem 7.8 (Product formula for exponential generating functions)

Let a_n be the number of ways to build a certain structure on an n element set, and let b_n be the number of ways to build another of structure. Let c_n is the number of ways to partition $[n]$ into two sets S, T, where we build the first structure on S and the second structure on T. If A, B, C are the corresponding exponential generating functions, then

$$C(x) = A(x)B(x).$$

Proof. If set S has i elements, there are $\binom{n}{i}$ ways to choose S, and $\binom{n}{n-i}$ ways to choose T. There are a_i ways to build on S and b_{n-i} ways to build on T, so

$$c_i = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

which corresponds to the formula we know. □

Recall that the Bell number b_n is the number of partitions of an n element set. In lecture 1, we had a recurrence for b_n but no closed form formula. Here, we will show that b_n has a nice exponential generating function,

$$B(n) = \sum_{n\geq 0} b_n \frac{x^n}{n!}.$$

Previously, we proved that

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} b_i,$$

where we considered the block containing element $n+1$.

We see that

$$\log B(x) = e^x + c$$

where c is some constant term. Here, it's okay to plug in $x = 0$, and we know that $b_0 = 1$, so $\log 1 = 0 = 1 + c$ and $c = -1$.

8 October 16, 2018

Last week we had a three-day week and an exam, so finally we're back!

8.1 Exponential generating functions, ctd.

Recall that the exponential generating function associated with sequence $(a_n)_{n\geq 0}$ is

$$\sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

For ordinary generating functions, we encoded information about compositions. The equivalent for exponential generating functions is set partitions.

We discussed that the product formula for A, B. If $C(x) = A(x)B(x)$, then $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. The coefficient c_n counts the number of ways to split a set into two parts, and build structure a on one part, structure b on the other. Similar to ordinary generating functions, we build two structures, but here we build them on sets, rather than compositions.

Last time we stopped at Bell numbers, where $B(n)$ is the number of set partitions of $[n]$. It was a lot of work, but we ended up with the following formula:

$$B(x) = \sum_{n\geq 0} B(n)\frac{x^n}{n!} = e^{e^x} - 1.$$

Today, we'll find an easier way.

In the world of ordinary generating functions, we often look at the quantity $1/(1 - A(x))$. Its counterpart in exponential generating functions is $e^{A(x)}$.

Recall that $S(n, k)$, the Stirling number of the second kind, is the number of partitions of $[n]$ into k blocks, and $B(n) = \sum_{k=1}^{n} S(n, k)$. For fixed k, let's look at the exponential generating function of $S(n, k)$,

$$\sum_{n\geq 0} S(n, k)\frac{x^n}{n!}.$$

1. For $k = 0$, $S(0, 0) = 1$ and $S(n, 0) = 0$ for all other n, so this function evaluates to 1.
2. For $k = 1$, the function is $\sum_{n\geq 0} \frac{x^n}{n!} = e^x - 1$.
3. Now the first non-trivial case. For $k = 2$, we have

   $$\sum_{n\geq 0} S(n, k)\frac{x^n}{n!} = \frac{1}{2} \cdot \left(\sum_{n\geq 1} \frac{x^n}{n!}\right)\left(\sum_{n\geq 1} \frac{x^n}{n!}\right)$$

   since we can consider the first and second sets, but they are indistinguishable (unordered). We take $n \geq 1$ because the sets should be non-empty.
4. For $k = 3$, we have three sets

   $$\sum_{n\geq 0} S(n, k)\frac{x^n}{n!} = \frac{1}{3!} \cdot \left(\sum_{n\geq 1} \frac{x^n}{n!}\right)^3.$$

5. The same argument tells us that in general, for k sets, we have

$$\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}\cdot\left(\sum_{n\geq 1}\frac{x^n}{n!}\right)^k.$$

Now we know that the Bell numbers are the sums of the Stirling numbers, so

$$\begin{aligned}\sum_{n\geq 0} B(n)\frac{x^n}{n!} &= \sum_{k\geq 0}\sum_{n\geq 0} S(n,k)\frac{x^n}{n!}\\ &= \sum_{k\geq 0}\frac{1}{k!}\left(e^x-1\right)^k\\ &= e^{e^k-1}.\end{aligned}$$

As a last note, we need to make sure that the inner function has no constant term, and e^x has constant term 1, so we take $e^x - 1$ instead.

Theorem 8.1

This is the **exponential formula** for exponential generating functions.

- Let a_n be the number of ways to build a certain structure on $[n]$.
- Let h_n be the number of ways to partition $[n]$ into non-empty subsets, and then build a structure of the first kind on each subset.
- By convention, we set $a_0 = 0, h_0 = 1$.

Let A, H be the exponential generating functions of a, h. Then

$$H(x) = e^{A(x)}.$$

Proof. If there are exactly k subsets, then $\frac{1}{k!}A(x)^k$ is the exponential generating function for the number of ways to build this structure. The general case is

$$H(x) = \sum_{k\geq 0}\frac{1}{k!}A(x)^k = e^{A(x)}$$

by Taylor expansion. □

Example 8.2 (Bell numbers)

Suppose $a_n = 1, \forall n \geq 0$ with $a_0 = 0$ (we build no structure, we just partition the sets).

Then we find that our previous answer was correct:

$$\begin{aligned}A(x) &= \sum_{n\geq 0} a_n\frac{x^n}{n!} = e^x - 1\\ H(x) &= e^{e^x-1}.\end{aligned}$$

Example 8.3

We have a Chinese restaurant with n people who sit around round tables. How many ways can they sit at the tables?

Let a_n be the number of ways n people can sit around a circular table, which is $(n-1)!$ (fix one person, permute the rest). Then

$$A(x) = \sum_{n \geq 1} (n-1)! \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n}$$

which is the power series of $-\log 1 - x$, derived below.

We plug in the exponential formula,

$$e^{-\log(1-x)} = \frac{1}{1-x} = \sum_{n \geq 0} n! \frac{x^n}{n!}.$$

So we have $n!$ ways. Sanity checking, we know that there is a bijection between cycles and one-line permutations, so $n!$ makes sense.

Remark. In high school we learned

$$-\log(1-x) = \sum_{n \geq 1} \frac{x^n}{n}.$$

Proof. Recall that $1/(1-x) = 1 + x + x^2 + x^3 + \dots$. We can integrate both sides,

$$\int \frac{1}{1-x} = \int 1 + x + x^2 + x^3 + \dots$$
$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + c$$

where $c = 0$ by plugging in $x = 0$. □

> "We could write ln, but none of this ln nonsense from high school. Real mathematicians write log."

Example 8.4

Let f_n be the number of set partitions of sizes 2,3, or 5.

The structure we build is

$$a_n = \begin{cases} 1 & n \in \{2, 3, 5\} \\ 0 & \text{otherwise.} \end{cases}$$

So $A(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!}$, and our answer is $\exp A(x)$.

Theorem 8.5
This is the **compositional formula** for exponential generating functions.

- Let a_n be the number of ways to build a certain structure on $[n]$.
- Let b_n be the number of ways to build another structure.
- Let g_n be the number of ways to partition $[n]$ into non-empty blocks, and then build a structure of the first kind on each blocks, then build a structure of the second kind on the set of blocks.
- By convention, we set $a_0 = 0, b_0 = 1, h_0 = 1$.

Let A, B, G be the exponential generating functions of a, b, g. Then

$$G(x) = B(A(x)).$$

We can think of a_n as operations on the individual elements within blocks, and b_n as operations on the blocks themselves.

The special case we saw earlier was $b_n = 1$ and $B(x) = e^x$.

Example 8.6 (Ordered set partitions)
What is the number of ordered sequence of subsets of $[n]$?

An ordered set partition imposes an order on the subsets: $(\{1,3\}, \{2,4,5\})$ is different from $(\{2,4,5\}, \{1,3\})$.

Let $a_n = 1$ for $n \geq 1$ and $a_0 = 0$ (same as before), so $A(x) = e^x - 1$. Let $b_k = k!$, since there are k ways to order k subsets. So

$$B(x) = \sum_{k \geq 0} k! \frac{x^k}{k!} = \frac{1}{1-x}$$

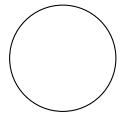
and $B(A(x)) = \frac{1}{2-e^x}$.

Example 8.7 (Compositions)
Let's find the number of compositions

oops fill in later

$$\sum_{k \geq 0} b_k \frac{A(x)^k}{k!}$$

Example 8.8
Let g_n be the number of ways that n people can form into non-empty lines and arrange the lines in circular order.



There are $a_n = n!$ ways to order people into a line, unless there are 0 people, in which case $a_0 = 0$. The generating function for a is $A(x) = \sum_{n\geq 1} x^n = x/(1-x)$.

Given these lines, there are $b_n = (n-1)!$ ways to order the lines into a circle. The generating function for b is

$$1 + \sum_{n\geq 1}(n+1)!\frac{x^n}{n!} = 1 + \sum_{n\geq 1} \frac{{}'x^n}{n} = 1 - \log(1-x).$$

Their composition is

$$\begin{aligned} G(x) = B(A(x)) &= 1 - \log\left(1 - \frac{x}{1-x}\right) \\ &= 1 - \log\left(\frac{1-2x}{1-x}\right) \\ &= 1 - \log(1-2x) + \log(1-x) \\ &= 1 + \sum_{n\geq 0}\frac{2x^n}{n} + \sum_{n\geq 0}\frac{x^n}{n} \\ &= (n-1)! \cdot (2^n - 1). \end{aligned}$$

This answer is very clean, so obviously there should be a combinatorial explanation, right? We can arrange people in a circle in $(n-1)!$ ways, and we select a non-empty subset of "heads" of lines from $2^n - 1$ possibilities.

Example 8.9

Find the number of permutations π of $[n]$ such that $\pi^6 = 1$.

If a cycle is too long, then it will not reach the identity within 6 rotations. That is, the cycle length must be $\in \{1, 2, 3, 6\}$. So

$$a_n = \begin{cases} (n-1)! & \text{if } n \in \{1,2,3,6\} \\ 0 & \text{otherwise.} \end{cases}$$

The exponential generating function is $A(x) = x + x^2/2 + x^3/3 + x^6/6$. We simply split up the set and form the cycles afterwards, so $G(x) = e^{A(x)}$.

Example 8.10

Find the number of derangements $D(n)$, or permutations without fixed points.

Derangements must not have fixed points, so

$$a_n = \begin{cases} (n-1)! & \text{if } n \geq 2 \\ 0 & \text{if } n = 0, 1. \end{cases}$$

The generating function is

$$A(x) = \sum_{n \geq 2} a_n \frac{x^n}{n!} = \sum_{n \geq 2} \frac{x^n}{n} = -\log(1-x) - x$$

where the final term subtracts out the 1 term. So

$$D(x) = \sum_{n \geq 0} xxx$$

Previously, we found a recurrence for the number of derangements, but we'll derive it again today!

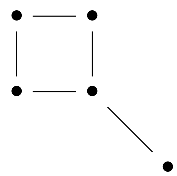
Today we looked at the compositional formula for exponential generating functions. This was similar to the compositional formula for ordinary generating functions, but we deal with set partitions in place of compositions. Arguably, set partitions occur more naturally than compositions, so exponential generating functions are a very powerful tool.

9 October 18, 2018

We've finished the enumeration half of this course, and today we will embark on a journey through graph theory.

9.1 Graph Theory

Definition 9.1. A **graph** $G = (V, E)$ is composed of a set of vertices and a set of edges, where each edge is a unordered pair of vertices.



For example, here we have 5 vertices and 5 edges.

Definition 9.2. A **simple graph** has no loops or multi-edges.[5]

- We say $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G.
- The **order** of G is written as $|G| = |V(G)|$.
- The number of edges is $e(G) = |E(G)|$.

In this class, most of the graphs we look at will be simple graphs.

Definition 9.3. A **multigraph** allows loops and multi-edges.

If edges are unordered pairs, we have an **undirected graph**. Otherwise, we have a **directed graph**. Conventionally, we write edges as (u, v) whether they are directed or not.

- We say that u, v are **adjacent** if $(u, v) \in E(G)$. This is equivalent to $u\tilde{v}$, and "u is a neighbor of v." Adjacency is *not* equivalent to connectivity.
- If $v \in e$, then vertex v and edge e are **incident**, or v is an endpoint of e.
- Two edges e_1 and e_2 are **incident** if they share an endpoint.
- Given $v \in V$, the set of neighbors of v, $N(v)$, is the **neighborhood** of v.
- The **degree** of vertex v is the size of its neighborhood, $|N(v)|$.
- A vertex is **isolated** if its degree is 0.
- A graph G is **d-regular** if all vertices have degree d.

[5] Multiple copies of each edge

Proposition 9.4

For $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2\,|E|\,.$$

Proof. We just count each edge twice, once for each endpoint. □

Example 9.5

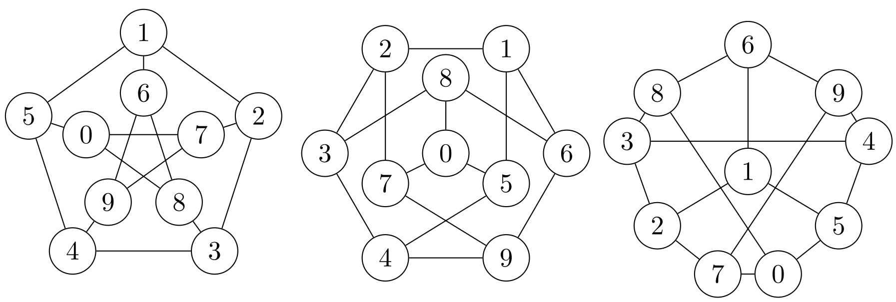
Is there a 3-regular graph on 7 vertices?

$\sum_v d(v) = 3 \cdot 7 = 21$ which is not divisible by 2, so this is not possible.

Definition 9.6. Given graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, an **isomorphism** is a bijection $\phi : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2, \forall u, v \in V$.

In colloquial terms, two graphs are isomorphic if there exists a labeling of each graph such that the sets of edges are the same.

A famous graph, for which many conditions fail, is known as the Petersen graph. It turns out that this graph has many non-obvious isomorphisms.

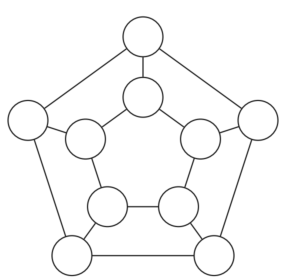


If someone gives you the labeling between two graphs, then we can check that they are equal in polynomial time, but we cannot give a polynomial time algorithm to decide whether they are isomorphic. Graph isomorphism is a problem in NP.

How do we prove that graphs are not isomorphic?

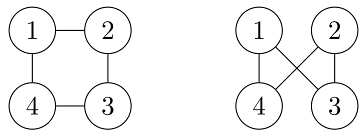
"I tried really hard and didn't find one"

This is not an answer. We need to find a property of one graph that does not hold for the other graph. How do we know this graph is not isomorphic to the Petersen graph?



This Petersen graph has no 4-cycles and this graph has 5 of them.

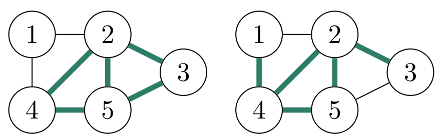
Generally when we talk about graph isomorphism, we refer to unlabeled graphs. In contrast, labeled graphs—usually vertex-labeled graphs—have a fixed labeling for vertices. Two labeled graphs are isomorphic if their labels correspond.



These graphs are isomorphic as unlabeled graphs but not as labeled graphs.

Definition 9.7. A graph $H = (U, F)$ is a **subgraph** of $G = (V, E)$ if $U \subseteq V$ and $F \subseteq E$.

- If $U = V$, then H is a **spanning subgraph**.
- If H contains all the edges of G between vertices in U, then we say that H is an **induced** subgraph. $G[U]$ is the subgraph induced by $U \subseteq V$.



induced spanning, not induced

There are lots of special graphs with names.

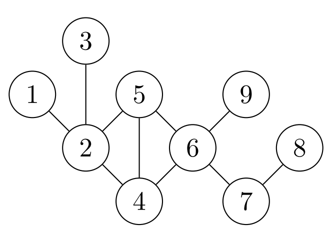
- The **complete graph** K_n includes all possible edges between n vertices.
- The **empty graph** on n vertices is just a set of vertices.
- A **bipartite graph** is a graph in which we can partition the vertex set $V = V_1 \cup V_2$ such that each edge has one endpoint in V_1 and the other endpoint in V_2.
- A **complete bipartite graph** $K_{m,n}$ is a bipartite graph in which all edges are drawn between the two sets of vertices, where $|V_1| = m, |V_2| = n$.

9.2 Walks, paths, and cycles

Definition 9.8. A **walk** in graph $G = (V, E)$ is a sequence of vertices $v_0, \dots, v_k$ such that $(v_i, v_{i+1}) \in E, \forall i = 0, 1, \dots, k-1$.

Definition 9.9. A **path** is a walk in which all the vertices are distinct.

Definition 9.10. A **cycle** is a path with $k \geq 2$ such that $(v_0, v_k) \in E$.



- 1254652523 is a walk.
- 12546 is a path.
- 2564 is a cycle.

As a matter of convention, we define the **length** of a path as the number of edges. This is a matter of controversy among graph theorists, some of whom claim that we should count vertices instead. In the above example, both the cycle and path have length 4.

Proposition 9.11

Every walk from u to v contains a path between u and v.

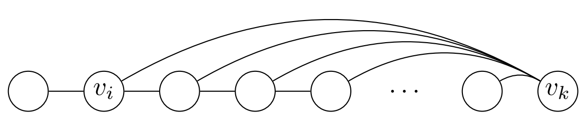
Proof. Let $u_0, u_1, \dots, u_k$, where $u_0 = u, u_k = v$, be a shortest walk between u and v using only edges in the original walk. We claim that this walk must be a path.

For contradiction, suppose not. There must exist two points in this walk, $i \le j$, such that $u_i = u_j$. Then $u_0, u_1, \dots, u_i, u_{j+1}, \dots, u_k$ is a shorter walk from u to v, which is a contradiction. □

Proposition 9.12

Every graph G with minimum degree $\delta \ge 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.

Proof. Let $v_0, v_1, \dots, v_k$ be a longest path in G. We claim that the neighbors of the final vertex is contained in the path; otherwise, we could extend the path.



We know that

$$\delta \le d(v_k) = |N(v_k)| \le k,$$

so the longest path contains at least δ neighbors of v_k, and thus has length at least δ.

Now let i be the lowest index such that $(v_i, v_k) \in E(G)$. It is easy to see that $|N(v_k)| \subseteq \{v_i, v_{i+1}, \dots, v_k\}$, so

$$\delta \le d(v_k) \le k - i$$

and the length of the longest cycle is $k - i + 1 \ge \delta + 1$.

□

10 October 23, 2018

10.1 Trees

Last time we gave a lot of definitions. Today we'll focus more on results from these definitions, but first, recall these ideas.

Definition 10.1. A graph G is **connected** if $\forall u, v \in V(G)$, there is a path from u to v.

Definition 10.2. A **connected component** is a maximal connected subgraph.

Remark 10.3. A "maximal" component cannot be made larger by adding additional elements, while "maximum" means the largest. The maximum is always maximal, but maximal components are not necessarily maximum.

Remark 10.4. A graph is **connected** if it has exactly one connected component.

- A single vertex is connected.
- The empty graph is not connected.

"This is the type of question where we ask, 'is 1 a prime?"'

Definition. A graph is **connected** if there exists a path between any pair of vertices, and a **connected component** is a maximal connected subgraph.

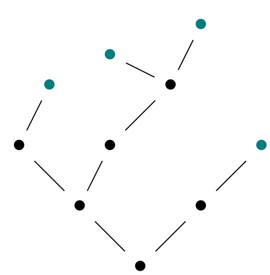
Proposition 10.5

A graph with n vertices and m edges has at least $n - m$ connected components.

Proof. We start with an empty graph and add in edges one at a time in any order. Observe that each new edge can reduce the number of connected components by at most 1. So with m edges, we have at least $n - m$ connected components. □

Definition 10.6. We start with a few definitions on trees.

- An **acyclic** graph has no cycles.
- A **forest** is an acyclic graph.
- A **tree** is a connected acyclic graph.
- A **leaf** is a vertex of degree 1 (marked in green).



Lemma 10.7

Every tree with at least 2 vertices has at least 2 leaves.

Proof. Consider a longest path in this tree. We claim that the endpoints of this path both have degree 1 in the tree. Consider an endpoint. It cannot have another neighbor outside the path, since this is a longest path. Likewise, it cannot have another neighbor inside the path, which would induce a cycle. Therefore, this endpoint has degree 1. We started with at least two vertices, so every path has at least 2 vertices, and there are at least 2 leaves. □

Lemma 10.8

Let G be a tree with n vertices and at least 2 leaves. If we delete a leaf, we obtain a tree on $n-1$ vertices.

Proof. We show that the graph is still connected and acyclic. If there are no cycles in the old graph, deleting a vertex will not induce a cycle.

Suppose we deleted vertex v. Now let $u, w \in V(G) \setminus \{v\}$. We show that there exists a path between every pair u and w that does not pass through v. Blah blah else degree 2?? □

There are some equivalent definitions of trees.

Theorem 10.9

Let G be a graph on n vertices. The following are equivalent.

1. G is connected and acyclic (our definition of a tree).
2. G is connected and has exactly $n-1$ edges.
3. G is acyclic and has exactly $n-1$ edges.
4. For every pair $u, v \in V(G)$, there is a unique path from u to v.

To prove this theorem, we prove a chain of implications; but first, let's start with a some definitions.

Definition 10.10. An edge of G is known as a **cut edge** if its deletion disconnects G.

[draw picture of diamond and c-shape connected by one edge)

Lemma 10.11

Any edge in a cycle is *not* a cut edge, since we can go around the cycle.

Proof. From statement (1), we show that (2) and (3) are true—that is, we show there are $n-1$ edges. We induct on n. If $n = 1$, there is only one vertex, so there are no edges, and the statements are trivially true. Now suppose $n > 1$.

Lemmas 10.7 and 10.8 provide a leaf v such that its deletion results in a graph that is acyclic and connected. The resulting graph has $n-1$ vertices with $n-2$ edges, so the original graph must have had exactly $n-1$ edges.

Now we start with (2) and show that (1) and (3) are true—that is, we show that our graph is acyclic. Starting from our graph G, we delete edges from cycles in G one by one until the resulting graph G' is acyclic. By lemma 10.11, G' is connected and acyclic, so by the (1) to (3), G' has exactly $n-1$ edges. Since G is acyclic, no edges were removed, so we started at G'.

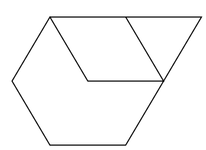
Now we start from (3) and show that (1) and (2) are true—that is, our graph is connected. Suppose G has k connected components, each with $n_1, n_2, \dots, n_k$ vertices. Each component is connected and acyclic. By (1) to (2), each component has $n_i - 1$ edges. The total number of edges is $e(G) = \sum_{i=1}^{n}(n_i - 1) = n - k$. However, we started with $n-1$ edges, so by lemma 10.5, we started with $k = 1$ connected components.

Now we show that (1) implies (4). Since G is connected, by definition there exists a $u \to v$ path. For contradiction, suppose G has two distinct $u \to v$ paths, P and Q. Let (x, y) be an edge on P but not Q. We can walk from x to y in G even without (x, y), since we can use the alternate path from $u \to v$. By the lemma from last time, there exists a $x \to y$ path in the graph G with edge (x, y) removed. If we add (x, y) back in, we complete a cycle in G, so G is not acyclic, a contradiction.

Finally, we show that (4) implies (1). It is trivial that a path implies connected, so we show that G is acyclic. If there is a cycle, then there are two equivalent paths between some $u, v \in V(G)$, so G must be acyclic.

□

Definition 10.12. Given a connected graph G, a **spanning tree** T is a subgraph of G which is a tree and contains all vertices of G.



Corollary 10.13

We give a few corollaries of theorem 10.9.

1. Every connected graph on n vertices has at least $n-1$ edges and contains a spanning tree. We simply delete edges until we have a spanning tree.

2. Every edge of a tree is a cut edge. If we start with a tree, there is only one path between every pair of vertices. If we delete an edge on that path, those vertices are no longer connected.

3. Adding an edge to a tree creates exactly one cycle. We generate a cycle because the edge we add creates two paths. This cycle is unique because if we remove that edge, then there will be two paths.

How many tree are there on n labeled vertices?

Theorem 10.14 (Cayley's formula)
The number of labeled trees on n vertices is exactly n^{n-2}.

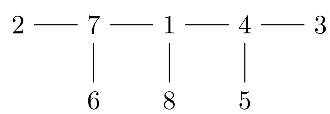
Proof. We prove this by Prüfer codes.

Given a labeled tree on n vertices, we generate a sequence $f(T)$ of length $n-2$ as follows.

1. Delete a leaf with the minimum label and append its neighbor to the sequence.

2. Stop when a single edge remains.

□



The Prüfer code of the graph is 744171.

The key observation is that the set of labels that do not appear in this sequence must be the leaves of T. Here, the missing numbers are $\{2,3,5,6,8\}$. The smallest missing number must have been attached to the first number in the code.

1. The first leaf removed was 2, which was attached to 7.

2.

Often, it is convenient to consider the complement of G, $\overline{G}$. That is, for each edge $(u,v) \in E(G)$, there does not exist edge $(u,v) \notin E(\overline{G})$.

With this idea, we can prove a bijection between cliques and independent sets.

Definition 10.15. A **clique** is a complete subgraph, and an **independent set** is a subset of vertices that induces an empty set.

Let $\omega(G)$ be the number of vertices of the maximum clique, and let $\alpha(G)$ be the number of vertices in the maximum independent set.

A clique in G corresponds to an independent set $\overline{G}$, so $\omega(G) = \alpha(\overline{G})$ and vice versa.

11 October 25, 2018

11.1 Trees, ctd.

Last time, recall that we ended at Cayley's formula for the number of trees.

Theorem

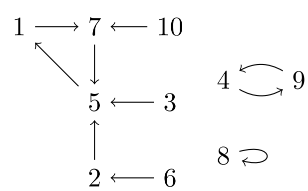
The number of labeled trees on n vertices is n^{n-2}.

We used Prüfer codes to give a bijection, but today we will show a more recent proof by Joyal. This will be a bijection from doubly rooted labeled trees on $[n]$ and sequences of length n with terms in $[n]$.

Proof. A doubly rooted tree is a tree with left root L and right root R, where L and R could be the same. The latter is equivalent to functions from $[n] \to [n]$. We start from a $f : [n] \to [n]$. For example,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}.$$

1. Form a directed graph (allow loops) with the edges $(i, f(i)), \forall i = 1, 2, \ldots, n$.

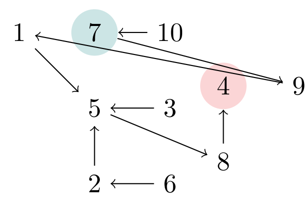


   Note that the out-degree of each vertex is 1.

2. Let M be the subset of vertices in some cycle (including loops). For example,
$$M = \{1, 3, 7, 4, 8, 9\}.$$

3. Sort the elements in M, $v_1 < v_2 < \cdots < v_k$. Note that f permutes M. All out-degrees are 1, so each vertex can only be involved in one cycle, and all cycles are directed (only move in one direction). Continuing along our example,
$$f = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}.$$

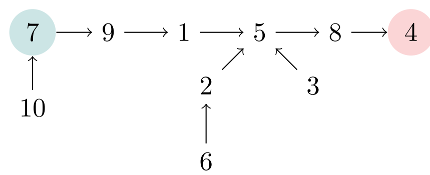
4. Remove all the edges in all cycles (including loops) and add path $f(v_1) \to f(v_2) \to \cdots \to f(v_k)$. Label $f(v_1)$ as left root L and label $f(v_k)$ as right root R. This path is the second line of the permutation above, $(7, 9, 1, 5, 8, 4)$.



5. Since trees are undirected, we drop all the directions.

Now we need to recover the tree.

We start with the path from L to R and add in directions. The path is completely L to R, so those directions are easy, but we also need to fill in the rest. Each vertex has out-degree 1, so we know where the rest point.



All vertices have out-degree 1, so the "cycle" path used up the out degree. Therefore, all the remaining edges must point "towards" the edge.

We need to recover the original cycles now. We know that the backbone path is the "second row" of the permutation, so the first row is just those numbers, in ascending order.

For vertices not in M, the graph tells us what f should be.

□

11.2 Connectivity

How connected is a graph? If you consider MIT's infinite corridor, it gets congested easily—there's only one way through. However, if you consider the Tiananmen Square, huge and open, it (should) be congested less with the same number of visitors. In some sense, the latter is more "connected" than the former.

Definition 11.1 (Vertex connectivity)**.** A **vertex cut** in a connected graph $G = (V, E)$ is a set $S \subseteq V$ such that $G \setminus S = G[V \setminus S]$ has more than one connected component.

In English, we refer to the mysterious quantity as "the graph obtained by deleting S and all edges incident to S" which is equivalent to "the graph induced on $V \setminus S$."

Definition 11.2. A **cut vertex** is a vertex v such that $\{v\}$ is a cut.

Definition 11.3. G is **k-connected** if $|V(G)| > k$ and $G \setminus X$ is connected, $\forall X \subset V$ with $|X| < k$.

- A 1-connected graph is the same as our usual "connected" graph.
- $\kappa(G)$ is the maximum k such that G is k-connected.

For example, the first graph can be disconnected by removing one vertex (the center), and the second graph can be disconnected by removing two vertices. The third graph is 3-connected.

- In general, the complete graph K_n has connectivity $k - 1$.
- The complete bipartite graph $K_{m,n}$ has connectivity m.

Proposition 11.4

$\kappa(G) \le \delta(G)$, where $\delta(G)$ is the minimum degree of G.

This means that we can remove $\delta(G)$ vertices to disconnect this graph.

Proof. If G is complete, $\kappa(G) = |V| - 1 = \delta(G)$, so we are good.

Now assume that G is not complete. Consider a vertex v of minimum degree $\le |V| - 1$ (else G is complete). We delete all the neighbors of G. Since G is not complete, we are still left with some neighbors not in $N(v)$, so the graph is disconnected. □

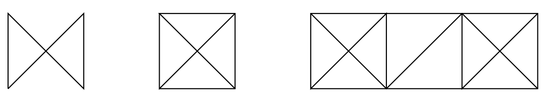
Remark 11.5. We *cannot* have high connectivity without a high minimum degree, but it is *not* true that a high minimum degree implies high connectivity. For example, consider two disjoint complete graphs.

Definition 11.6 (Edge connectivity)**.** In a connected graph G, a **disconnecting** set of edges is a set $F \subseteq E(G)$ such that $G \setminus F$ has more than one component.

Let $[S, T]$ be the set of edges with one endpoint in S and the other in T. The **edge cut** is an edge set of the form $[S, \overline{S}]$ for some $\emptyset \ne S \subsetneq V(G)$.

A graph G is **k-edge connected** if every disconnecting set of edges has size at least k.

We denote $\kappa'(G)$ as the edge connectivity of G, or the maximum k such that G is k-edge connected, also the minimum number of edges we must remove to disconnect G.



$\kappa = 1$, $\kappa' = 2$ $\kappa = 3$, $\kappa' = 3$ $\kappa = 2$, $\kappa' = 3$

Theorem 11.7

$\kappa(G) \le \kappa'(G) \le \delta(G)$.

Proof. First we show that $\kappa'(G) \le \delta(G)$. Let $v \in V(G)$ be a vertex of minimum degree. If we remove all edges incident to v, then we have disconnected the graph.

Now we show that $\kappa(G) \le \kappa'(G)$. That is, it is possible to find a vertex cut of size at most $\kappa'(G)$. Let S be an edge cut. If every vertex in S is adjacent to every edge in $\overline{S}$. Then the conne

$$\kappa'(G) = |S|\,(|V| - |S|) \ge |V| - 1.$$

By definition, $\kappa(G) \le |V| - 1$.

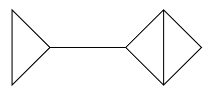
The second case is that $\exists x \in S, y \in \overline{S}$, where x, y are not incident.

Ugh he's lost me while drawing diagrams ripu time for pictures

$|V| \le \left|[S, \overline{S}]\right|$, which proves the theorem ???? □

> "You can think of S and $\overline{S}$ as US and Canada. A cut would close off the border."
>
> "US and Mexico would be a better example. [laughter from class]"
>
> "[pause] One has to be politically correct these days. I'm Canadian, so I can make Canadian jokes."

Definition 11.8. A **block** of a graph is a maximal connected subgraph of G that has no cut vertex.



In the third diagram, each little diamond is not a block, because it is not maximal. There is only one block.

Remark 11.9. A block with at least 3 vertices is 2-connected. An edge is a block if and only if it's a cut edge.

Proposition 11.10

Every pair of blocks share at most one vertex.

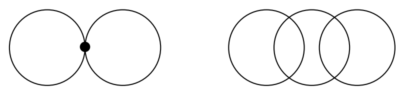
Sketch of proof. Suppose otherwise, and a pair of blocks share two vertices. Then their union has no cut vertices (the only possible cut vertices would be the intersection, which is greater than one vertex). So we can extend the blocks and they are not maximal. Thus, we have proved this proposition. □

12 October 30, 2018

12.1 Connectivity, ctd

Recall that a graph is k-connected if it has more than k vertices (only relevant for complete graphs), and it remains connected whenever we remove fewer than k vertices. Intuitively, a clique is more connected than a path: we only need to remove one vertex to destroy a path.

- A 1-connected graph is just our classic notion of "connected."
- A 2-connected graph is built from cycles.



Alternatively, there are multiple ways to walk somewhere.

Theorem 12.1 (Whitney's theorem)
Let G be a graph with at least 3 vertices. Then G is 2-connected if and only if the following holds: every pair of distinct vertices is joined by two internally disjoint paths (paths are disjoint except at start and end vertices).

Proof. We start with the "if" direction: if every pair of distinct vertices is joined by two internally disjoint paths, then G is 2-connected.

For $w \in V(G)$, if we delete w from G, then $G' = G \setminus \{w\}$ is still connected. Let u, v be a pair of vertices in G'. There exists two internally disjoint paths from $u \rightsquigarrow v$, so if w removes one path, the other still exists.

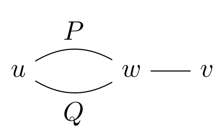
Now we show the "only if" direction. Let G be 2-connected and let u, v be distinct vertices. We induct on the distance $d_{u,v}$ between u, v, or the length of the shortest path between u and v.

The base case is $d_{u,v} = 1$, or (u, v) is an edge. The first path is just the edge (u, v). To find another path, we would like to "destroy" this first path. Recall from last time, that the edge connectivity is at least as large as the vertex connectivity, so G is 2-edge-connected. Thus, if we remove edge (u, v), G remains connected, and there is another path.

Now we induct on $k = d_{u,v} > 1$.

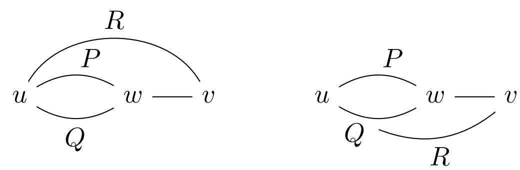
> "What is the verb form of 'induction'? Induce. But for some reason, no one says 'induce.' We always say 'induct.' "—yufeiz

Since G is connected, there is at a shortest path from $u \rightsquigarrow v$. Let w be the vertex adjacent to v on the path. The distance $d_{u,w} = k - 1$. By the inductive hypothesis, we can find two internally disjoint $u \rightsquigarrow w$ paths P, Q.



If $v \in P$ or $v \in Q$, then we are done, since they are internally disjoint. Otherwise, if we delete w, there must be another path R from u to v in G.

If R is internally disjoint from P, Q, then we are done. Otherwise, let x be the last vertex on R that lies on P or Q, other than u, v. Without loss of generality, x lies on Q. So we take $P + u$ and $u \xrightarrow{Q} x \xrightarrow{R} v$.



"It's like trying to fit a carpet in a room that's too small."—yufeiz

□

Corollary 12.2

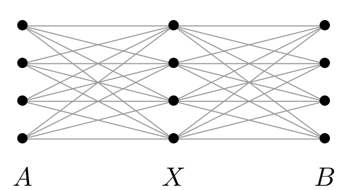
Let G be a 2-connected graph with at least 3 vertices. Every pair of distinct vertices in G lie on a common cycle.

Proof. In one direction, if there are two paths, then we connect them to create a cycle. In the other direction, u, v lie on a cycle, so deleting one path does not disconnect the graph. □

We have shown that we can remove vertices and remain 2-connected, but this idea also translates to general k.

Definition 12.3. Let $A, B \subset V$. An $A - B$ path is a path with one endpoint in A and the other in B, and all interior points lie outside $A \cup B$. Any vertex in $A \cap B$ is a trivial $A - B$ path of length 0.

Let $X \subset V$ (or $X \subset V$). We say that X separates A and B if every $A - B$ path in G contains a vertex (or edge) in X.



Theorem 12.4 (Menger's theorem (1927))

Let $G = (V, E)$ be a graph and let $S, T \subset V$. Then the maximum number of disjoint $S - T$ paths is equal to the minimum size of an $S - T$ separating set.

Remark 12.5. This idea is similar to max-flow min-cut, but here we work with vertex flow, so it's a "vertex version" of max-flow min-cut. Ford-Fulkerson is a generalization of this idea.

> "I remember taking 6.046 when I was an undergrad. . . I read somewhere that max-flow min-cut was derived independently, since the Soviets were trying to find the maximum flow through their railroads, and the Americans were trying to block those railroads."—yufeiz

Remark 12.6. A separating set is allowed to intersect S and T.

We introduce the idea of edge contractions. Suppose G contains edge $e = (u, v)$. The $G \setminus e$ is the graph obtained by merging u and v into the same vertex and deleting duplicate edges.

If there are k $S - T$ paths, then every $S - T$ separating set contains at least k vertices. That is, we must delete at least k vertices.

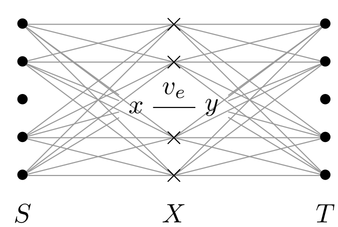
We induct on the number of edges. If $e(G) = 0$, then we have a set of vertices and no edges, and all $S - T$ paths are vertices in $S \cap T$. So the maximum number of disjoint $S - T$ paths is $|S \cap T|$, and the minimum separating set S, T is also that exact intersection.

Now suppose $e(G) > 0$. Let $(x, y) \in E(G)$.

Suppose we contract (x, y) to vertex v_e in graph G'. By the inductive hypothesis, our claim is true for this altered graph. Now suppose we expand this edge again. If no path goes through v_e, then we have our disjoint paths still, and we are done. On the other hand, if some path goes through v_e,

By the inductive hypothesis, the result is true for G'. Here we include $v_e \in V(G)$ in the new S' if either of the endpoints lie in S, and likewise for T.

By induction, there exists k disjoint $S - T$ paths in G and a k-vertex set X separating S from T in $G' = G - e$.



If $v_e \notin X$, then X still separates S from T in G, and we have the desired paths and separating set. So we assume that $v_e \in X$. X is $S - T$ separating in G'. Every $S - X$ separating set in G' is a $S - T$ separating set in G. We apply induction to find k disjoint $S - X$ paths in G'. Similarly, we find k disjoint $X - T$ paths in G'.

So we've round

Apparently this proof can be found in Diestel.

13 November 1, 2018

The professor starts by noting common mistakes on our most recent problem set.

Example

Given a connected graph G, G is a tree if and only if every family of pairwise intersecting paths share a common vertex.

One direction is easy—if a graph is not a tree, it has a cycle, which we can break up into pairwise intersecting paths without a common vertex.

The other direction is tricky. Even if P_1, P_2, P_3 intersect at different vertices, they mind not intersect at the endpoints, so you can't just claim that they form a triangle.

Instead, consider these arguments.

> Suppose P_1, P_2 intersect. Then their intersection $P_1 \cap P_2$ must be a path. Likewise, for path P_3, $P_1 \cap P_3$ is a path.

Here is a second solution.

> Suppose we remove a leaf from T. If the leaf is in every path, it is in all paths. Otherwise, it is not in any path, and we can remove it, then induct on the number of vertices.

13.1 Connectivity, ctd.

Last time we introduced Menger's theorem. Today we will discuss it a bit more.

Theorem (Menger's theorem (1927))

Let $G = (V, E)$ be a graph and let $S, T \subset V$. Then the maximum number of disjoint ST paths is equal to the minimum size of an ST separating set.

Sketch of proof. Suppose we wanted to show that a set of ST paths is maximal. We could provide a collection of ST paths, along with a separating set, with one vertex on each path. Then we can show that there aren't any more.

In one direction, we need at least one vertex to destroy each path.

Let k be the minimum number of vertices needed to separate S and T. We show that there exists k disjoint ST paths. Fix an edge, $e = (x, y)$ and consider the contraction $G - e$. By the inductive hypothesis, the contracted graph has ST separating set Y with size $|Y| < k$. Suppose that there do not exist k disjoint ST paths. If we find k disjoint ST paths in $G - e$, then we can expand these paths to get ST paths in G.

Since X is separating, the SX and XT paths are disjoint, so we can connect them. Thus we have found k disjoint ST paths.

Blah blah

□

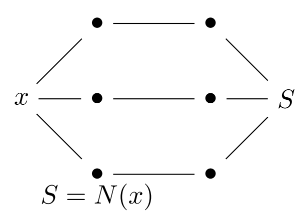
Now we will explore the consequences of Menger's theorem.

Menger's theorem is slightly awkward—it has a set of start points and a set of end points. What if we just cared about a single start s and end t? How many paths are there from $s \rightsquigarrow t$? In Menger's theorem, we do not include paths that share start and end vertex. Instead, we can just apply Menger's theorem on the neighborhoods of s and t.

Corollary 13.1 (Fan lemma)

Let $G = (V, E)$ be a graph. Let $S \subset V$ and let $x \in V \setminus S$. Then the minimum number of vertices in $V \setminus \{x\}$ to separate x from S is equal to the maximum number of xS paths, disjoint except at x.

Proof. We just apply Menger's theorem to the neighborhood of x.



□

Corollary 13.2

Let $u, v \in V(G)$ be distinct vertices where $(u, v) \notin E(G)$. Then the maximum number of internally vertex disjoint $u \rightsquigarrow v$ paths is equal to the minimum size of a set of vertices (other than u, v) separating u, v.

Proof. Let $S = N(u), T = N(v)$. Then we are done. □

Remark 13.3. We can also deduce Menger's theorem from these lemmas. We just connect every vertex $x \in S$ to a start vertex s and every vertex $y \in T$ to end vertex y.

Corollary 13.4

Suppose u, v are distinct vertices. Then the maximum number of edge disjoint $u \rightsquigarrow v$ paths is equal to the minimum number of edges separating u, v.

Proof. The line graph of G is denoted $L(G)$. The vertices of $L(G)$ are the edges of G. Two edges in $L(G)$ are adjacent if they share a vertex—that is, if they are incident in G.

We apply Menger's theorem to the line graph $L(G)$, where S is the set of edges of G incident to u, and T is the set of edges of G incident to v. Each sequence of vertices in $L(G)$ is a sequence of edges in G. □

Theorem 13.5 (Global version of Menger's theorem)
Let $G = (V, E)$ be a graph.

1. G is k-connected if and only if there exist k internally vertex disjoint paths between every pair of distinct vertices in G. For $k = 2$, this is equivalent to Whitney's theorem.

2. G is k-edge-connected if and only if there exist k internally edge disjoint paths between every pair of distinct vertices in G.

Proof. These follow easily from our corollaries.

1. This almost follows from 13.2, but we need to be careful when $(u, v) \in E$.

   In the $\Leftarrow$ direction, if there are k internally disjoint paths, then we need to delete a vertex from each path.

   In the $\Rightarrow$ direction, if $(u, v) \notin E$, the result follows from corollary 13.2. Otherwise if $(u, v) \in E$, consider $G' = G - (u, v)$. Suppose there are not k internally disjoint paths. Then we cannot find $k - 1$ internally disjoint paths in G'. By corollary 13.2, we find that G' contains a u, v separating set X of size at most $k - 2$. By definition, $|V| \geq k + 1$, so there exists a vertex $w \in V \setminus (\{u, v\} \cup X)$. Either w is not connected by a path to u or v. Without loss of generality, suppose w is not connected to u.

   Then $X \cup \{v\}$ separates u and w in G, which is a set of size at most $k - 1$. This set contradicts the hypothesis that G is k-connected. Therefore, there must be k internally disjoint paths from u to v.

2. This follows from corollary 13.4.

□

"This is mostly for cultural value."—yufeiz

Suppose we wanted to consider directed graphs too. We would have four versions of Menger's theorem,

$$\{\text{vertex, edge}\} \times \{\text{undirected, directed}\}.$$

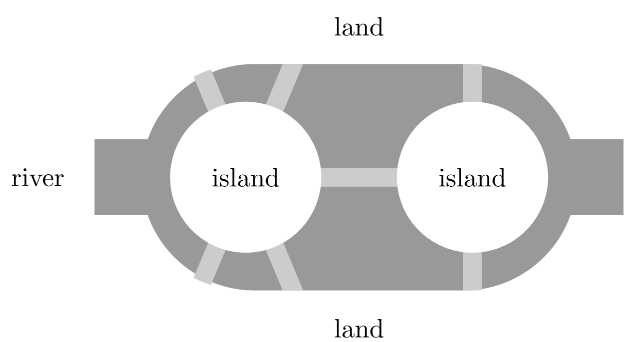
14 November 6, 2018

The midterm is on Thursday. The professor will be holding office hours today. It is suggested that we study the practice exam and solutions.

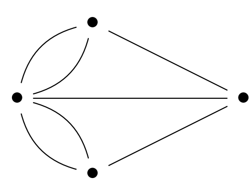
> "The TA was very generous in giving out points, so if you got a 7/10, you did not solve the problem."—yufeiz

14.1 Eulerian tours and Hamiltonian cycles

Today, we come to the origins of graph theory. In a German city known as Königsberg, there was a city with many bridges. A common question at that time was—is it possible to walk all seven bridges without repeating?



Euler came along and proved that no, it is not possible to walk through the graph passing through every edge exactly once.



Why? Let's think about degrees. Except for a start and end vertex, all intermediate vertices in this walk must have even degree. All edges are paired up because we enter a vertex and leave a vertex. In the Königsberg example, there are four vertices of odd degree, so it is not possible to walk that graph. Let's formalize this understanding.

Recall that in a walk, we are allowed to repeat vertices and edges.

Definition 14.1. A **trail** is a walk with no repeated edges.

Intuitively, if you think about a hiking trail, you can visit the same location more than once, but it's really boring to repeat the same segment of a hike.

Definition 14.2. A **Eulerian trail** is a walk that passes through every edge exactly once.

Definition 14.3. An **Eulerian tour**, also known as an Eulerian circuit, is an Eulerian trail that starts and ends at the same vertex.

Theorem 14.4

A connected (multi)graph has an Eulerian tour if and only if every vertex has even degree.

Proof. We start from the $\Rightarrow$ direction. If we have an Eulerian tour, then all degrees are even since at every vertex, in this walk, the edges going into the vertex are paired with the edges going out.

The $\Leftarrow$ direction is more interesting. It is wrong to randomly walk, because not all walks form an Eulerian trail—in particular, we may not reach all edges.

So let's try this strategy instead. Consider a longest trail. This trail must start and end at the same vertex, since our graph is event, and we can extend the trail by adding the degree going out from the end.

Now we claim that this trail must include all the edges. Suppose not. Since the graph is connected, there exists an edge e not on the trail but is incident to the trail. Then we can start with e and produce a longer trail. □

Corollary 14.5

A connected multigraph G has an Eulerian trailif and only if it has 0 or 2 vertices of off degree.

Proof. The $\Rightarrow$ direction is simple: all the intermediate vertices must have even degree.

Now we show the $\Leftarrow$ direction. If two vertices have odd degree, we can simply make a new edge between the two vertices of odd degree. □

Eulerian tours are so easy that they give it as a grade school problem! But a small change to the problem, and it becomes much harder.

Definition 14.6. A **Hamiltonian path** is a spanning path (i.e. a path that contains all vertices).

Definition 14.7. A **Hamiltonian cycle** is a spanning cycle.

We say that a graph G is Hamiltonian if it contains a Hamiltonian cycle. For example, the skeleton of a cube is Hamiltonian.

As a historical side note, the mathematician Hamilton came up with the puzzle known as a traveler's dodecahedron.

It turns out that deciding whether G is Hamiltonian is NP-complete. However, we can still provide some necessary and sufficient conditions on G.

Proposition 14.8

If G is Hamiltonian, then for all subsets $S \subseteq V(G)$, $G \setminus S$ has at most $|S|$ connected components.

For example, when $|S = 1|$, [insert pic]

Proof. Suppose G has a Hamiltonian cycle. Let $C_1, \ldots, C_k$ be components of $G \setminus S$. Imagine walking the Hamiltonian cycle in some direction. Let v_i be a vertex encountered immediately after leaving C_i. Note that $v_i \in S$. Furthermore, all v_i are different. Since there are k components, $|S| \geq k$.

□

While proposition [todo] is a necessary condition, it is not sufficient.

For example, the windmill graph satisfies proposition [todo] but it is not Hamiltonian.

This is because each of the blue vertices must be in the cycle, but the only two options for going in and out are the highlighted edges, and we must traverse the center twice.

Corollary 14.9

If a connected bipartite graph $G = (V, E)$ is Hamiltonian, and the bipartition is $V = A \cup B$, then $|A| = |B|$.

Proof. If $|A| \leq |B|$, then we delete A and B becomes isolated vertices. By theorem [todo], $G \setminus A$ cannot have more than $|A|$ connected components. □

Now we will give a sufficient condition.

Theorem 14.10 (Dirac's theorem)

Let G be a graph on at least 4 vertices. If all degrees are at least $n/2$then G is Hamiltonian.

Note that this is not necessary, since we could just have a simple cycle.

Proof. G is connected since otherwise, one component has fewer than $n/2$ vertices, so the degree is strictly less than $n/2$ somewhere.

Now we use our favorite trick, which is "consider the longest path." Let $v_0, v_1, v_2, \ldots, v_k$ be the longest path. Then we cannot extend it further, so all the neighbors of v_0, v_k are contained in the path. By pigeonhole principle, since $k < n$, there exists some i such that $(v_0, v_{i+1}), v_i, v_k) \in E$. This gives us a cycle through the k vertices.

This is a trick known as "rotation extension" —which "sounds like physical therapy"

If this path does not contain all vertices of G, then we can find an edge e with one endpoint on the cycle and the other endpoint not on the cycle. Then we can extend the original path by starting from this new edge and going around the cycle. Therefore, our original path must contain all the vertices in G, and we have found a Hamiltonian cycle. □

Theorem 14.11

If G on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for al non-adjacent $u, v \in V(G)$, then G is Hamiltonian.

That is, in the step where we use pigeonhole principle, the degrees need only add up to n. This is a weaker condition.

Let's go back to the Petersen graph. Is the Petersen graph Hamiltonian?

There are 5 inner vertices and 5 outer vertices. We must pass between the inner and out rings either 2 or 4 times.If there are 2 passes,

15 November 13, 2018

15.1 Matchings

Matchings occur naturally in many fields. In this class, we will focus on the graph theory applications, but these are not the only cases.

- Traditionally, we could pair men and women (marriage problem) or, in today's forward society, just people (roommate problem).
- We could also match students and schools, since schools have finite capacity. This is a many-to-one assignment, but we can duplicate the schools for a one-to-one assignment.
- Similarly, we could assign medical residents to hospitals (stable matching problem) There are preferences on both sides, but we would prefer if no two residents want to switch with each other.
- We could also schedule classes and rooms. For simplicity, assume all classes are hour blocks. We have a bipartite graph, with classes on one side and rooms with times on the other.
- Suppose we want to deliver mail along a graph, where the edges are streets (Chinese postman problem).

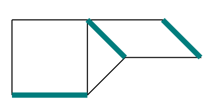
  > "If you look on Wikipedia, it has a more culturally sensitive name called the 'route inspection problem.' "—yufeiz

  If the graph is Eulerian, we can visit every edge without repeats, and this is optimal. Otherwise, we are concerned about odd-degree vertices. Suppose we add an edge between every pair of odd-degree vertices (u, v), where the length is the shortest distance between u and v.

  Now we find a minimum-weight perfect matching between vertices in this graph. We can convince ourself this is the best we can do.

Matchings are everywhere, and they will be the focus of our next two lectures.

Definition 15.1. A matching in graph G is a subset of edges with no two edges sharing a vertex.



A question we may ask is, given graph G, what is the size of the largest matching?

Definition 15.2. A **perfect matching** is a matching that covers every vertex.

For example, the Petersen graph has a perfect matching—the edges between the inner and outer ring.

Remark 15.3. Non-bipartite matchings are less common in applications, and their proofs are much more difficult.

Theorem 15.4 (Hull's theorem)
Let G be a bipartite graph with vertex bipartition $A \cup B$. Then G has a matching covering A if and only if

$$|N(S)| \geq |S|, \forall S \subseteq A.$$

This property is *necessary* since if this property does not hold for some set S, then all the vertices in S cannot be matched.

> "I don't care what happens on your computer; I just want the result. Happens in most things in life."—yufeiz

Hull's theorem gives us a means to certify that no matching exists: we simply produce the S for which the condition fails. For a bit of culture, this problem is in coNP since we have a certificate of failure.

Proof of Hull's theorem. The only if direction $\Rightarrow$ is obvious.

The if direction $\Leftarrow$ is more interesting. We induct on $|A|$. When $|A| = 1$, we either have an edge or we don't. Now suppose $|A| \geq 2$.

Case 1 We have some "room to spare." That is,

$$|N(S)| \geq |S| + 1, \forall \emptyset + S \subsetneq A.$$

So we arbitrarily match some $a \in A, b \in B$. Then we claim that the above equation holds for $G - a - b$. By induction, $G - a - b$ has a matching covering $A - a$, so we add in edge (a, b) to get a matching covering A.

Case 2 We have no room to spare. So there is some $S \neq \emptyset, A$ such that $|N(S)| = |S|$. In this case, [...]

Suppose we split G in two, such that G_1, G_2, etc...

We claim that Hull's condition is satisfied on G_1, G_2.

Uhh got lost lol reading piazza for 867

□

Corollary 15.5
In a bipartite graph G with bipartition $A \cup B$, if there exists some integer d such that

$$|N(S)| \geq |S| - d, \forall S \subset A,$$

there exists a matching of size at least $|A| - d$.

Proof. Let us modify G as follows. Add d vertices to B connected to all vertices of A. This new graph G' satisfies the condition from Hull's theorem, since

$$|N_{G'}(S)| \geq |N_G(S)| + d \geq |S|.$$

By Hull's theorem, there exists a matching covering A in G'. We remove the new vertices to obtain a matching in G of size at least $|A| - d$. □

Corollary 15.6

Let $k \geq 1$. Every k-regular bipartite graph has a perfect matching.

Proof. We start with a k-regular bipartite graph G with bipartition $A \cup B$. Let $S \subset A$.There are $|S| \cdot k$ edges emanating from S. On the right side, there are at most $k \cdot |N(G)|$ edges incident to B. Therefore, $|N(S)| \geq |S|, \forall S$, and Hull's theorem guarantees us a perfect matching. □

This is not true for non-bipartite graphs. For example, consider the triangle.

Corollary 15.7

Every k-regular bipartite graph is the union of k perfect matchings.

Proof. We apply corollary 15.6 to find a perfect matching and remove it, to obtain a $k - 1$ regular graph. Then we repeat. □

Definition 15.8. A **k-factor** is a k-regular spanning subgraph.

- A 1-factor is the same as a perfect matching. So if a graph has a perfect matching, it is "1-factorizable."
- A 2-factor is a spanning set of disjoint cycles.

Corollary 15.9

Let $k \geq 1$. Every $2k$-regular graph (not necessarily bipartite) has a 2-factor.

Proof. Assume that G is connected, else we can look at the connected components.

Since the graph is $2k$-regular, every vertex has even degree, and we can find an Eulerian tour. If we assign a direction to the tour, there are k edges going into v and k edges going out of v. For every vertex $v \in V(G)$, split the vertex into $v_{\text{in}}, v_{\text{out}}$, each of which takes on the edges going in or out.

Note that this new graph G' is bipartite with $V_{\text{in}} \cup V_{\text{out}}$: all vertices going out point to vertices going in, and vice versa. This bipartite graph is k-regular, so there is a perfect matching in G'.

If we collapse $v_{\text{in}}, v_{\text{out}}$ back into G, we reduce the perfect matching into a 2-factor. □

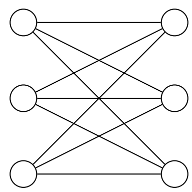
16 November 20, 2018

16.1 Planarity

Today let's start with a motivating example.

Example 16.1

There are three utility companies (gas, electricity, water) and three houses. Can we connect the utilities and houses without crossing?



these cross!

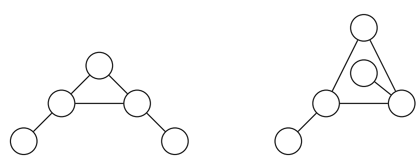
We see that this is not possible.

Definition 16.2. A graph G is **planar** if it can be drawn in the plane using continuous curves as edges and no two edges cross.

Remark 16.3. If we want to be rigorous, we need to delve into topology, but we won't.

> "So if we have a piece of paper… who wants to sacrifice their homework?"—yufeiz

- A specific drawing of G is called a **planar embedding** of G, or simply a **drawing**.
- The same graph can have multiple different drawings.



Definition 16.4. A **plane graph** G is a planar graph with a specific drawing.

Theorem 16.5 (Fary's theorem)

Every plane can be redrawn using line segments as edges.

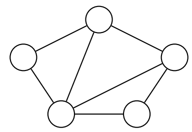
Theorem 16.6 (Circle packing theorem)

Every plane graph can be realized as the tangency relation between disjoint disks in the plane.

Proposition 16.7

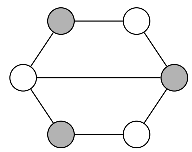
We cannot draw K_5 or $K_{3,3}$ on the plane.

Proof. K_5 has a 5-cycle, so we embed the 5-cycle.



There are 3 more edges, either all inside or all outside. However, we can only fit 2 of them.

$K_{3,3}$ has a 6-cycle.



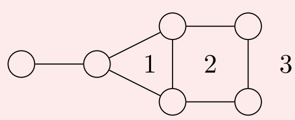
We cannot add edges within the two halves, so we cannot fit the remaining edges. □

Theorem 16.8 (Euler's formula)

Let G be a connected plane graph with exactly v vertices, e edges, and f faces. Then $v - e + f = 2$.

Example 16.9

For example, this graph has three faces: the triangle, the square, and the outside, which always counts!



Proof. First, we check that the result holds for trees. In a tree, there are $v - 1$ edges with 1 face, so Euler's formula holds for trees.

Otherwise, there is a cycle. Select any cycle and remove an edge. The graph is still connected, but the number of edges decreases by 1, and the number of faces decreases by 1 (we broke the cycle). If Euler's formula holds for the smaller graph, it holds for the bigger graph (induction). □

Theorem 16.10

A planar graph G on $n \geq 3$ vertices has $\leq 3n - 6$ edges.

Proof. First assume that G is connected; otherwise we add edges to G until it is connected, while maintaining planarity.

Every face has ≥ 3 edges (we assume that $n \geq 3$). If we sum over all faces, we have $\geq 3f$ edges, but every edge is counted exactly twice. So we find that $2e \geq 3f$ and $f \leq 2e/3$. By Euler's formula,

$$n - e + \frac{2}{3}e \geq 2$$
$$e \leq 3n - 6.$$

□

Theorem 16.11

If a plane graph G has at least 3 vertices and is triangle-free, then $e(G) \leq 2n - 4$.

Proof. Every face has at least 4 edges on its boundary, so using the same method, we find that $2e \geq 4f$ and $e \leq 2n - 4$. □

We know that $K_{3,3}$ avoids triangles since it is bipartite, so $e(G) \leq 2n - 4 = 8$. 8 is not enough to connect all pairs of vertices.

Remark 16.12. The Petersen graph is not planar.

Proof. The **girth** of a graph is the minimum cycle length. The girth of the Petersen graph is 5. So $2e \geq 5f$ and $n - e + 2e/5 \geq 2$. □

Let's take an interlude and talk about platonic solids.

Definition 16.13. A **platonic solid** is a regular polytope, or a shape where all faces are congruent regular polygons and there is the same number of faces at every vertex.

For example, we may have heard of the cube, tetrahedron, octahedron, dodecahedron, and icosahedron. These are, in fact, the only platonic solids.

"The ancient Greeks thought these were mythical creatures."—yufeiz

Proof. If we start with a planar graph on a plane, we can similarly draw it on a sphere. Likewise, if we draw a planar graph on a sphere, we can "poke a hole" and unfold the sphere into a plane.

We can apply Euler's formula to platonic solids. Suppose all faces are regular k-gons and every vertex is incident to m faces. Each vertex is incident to m edges, so $mv = 2e$.

On the other hand, we can count edges through the faces, so there are kf edges. We solve to find that $v = 2e/m$, $f = 2e/k$, and through Euler's formula,

$$\frac{2e}{m} - e + \frac{2e}{k} = 2$$
$$\frac{1}{m} + \frac{1}{k} = \frac{1}{2} + \frac{1}{e}.$$

We restrict m, k, e to the natural numbers. The only solutions are

$$(k, m) \in \{(3,3), (3,4), (3,5), (4,3), (5,3)\}.$$

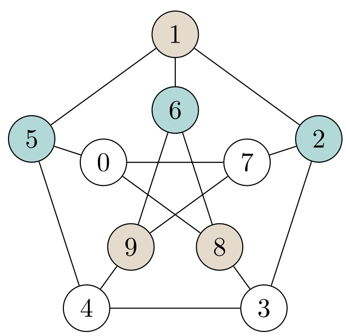
□

We might ask—are there any other graphs like $K_{3,3}$ or K_5? We could, for instance, split each edge of K_5 into several segments. This is known as a K_5-subdivision, which is not planar. More generally, an H-subdivision starts with H and replaces every edge by a path, each of which is disjoint from each other, except at the endpoints.

Furthermore, if any graph G contains a K_5 subdivision or a $K_{3,3}$ subdivision, then G is not planar.

Theorem 16.14 (Kuratowski's theorem)
A graph is planar if and only if it does not contain any $K_{3,3}$ or K_5 subdivisions as subgraphs.

The Petersen graph has a $K_{3,3}$ subdivision.



Definition 16.15. H is a **minor** of G if we can obtain H from G through edge deletions, vertex deletions, and edge contractions.

Subgraphs are minors where we only delete vertices.

Remark 16.16. Taking minors preserves planarity.

In the Petersen graph, if we contract the edges between the inner and outer ring, we end up with K_5.

Theorem 16.17 (Wagner's theorem)
A graph is planar if and only if it has no K_5 and $K_{3,3}$ minors.

As mathematicians, let's take a step further. Which graphs can we embed on a torus? Or a "torus with a handle"? (as dubbed by yufeiz) We can get a torus from a sheet of paper by rolling it up and gluing the two ends.

We can embed $K_5, K_7, K_{3,3}$ on a torus, but not K_8. Is there a general theory for this?

Theorem 16.18 (Graph minor theorem)
Every minor closed family has a finite list of excluded minors.

That is, if a property is preserved under taking minors, then every family has a finite number of obstructions.

This is the deepest theorem in graph theory to date, proved over 20 years, over 20 papers and 500 pages.

Today, we do not know the exact list of graphs that cannot be embeded on a torus, but it is at least 1600 long (and finite).

17 November 27, 2018

Next we have the last problem set due next Tuesday and midterm 3 next Thursday. The midterm will cover up to colorings, which will be discussed this week. Next Thursday be the final class.

17.1 Colorings

Definition 17.1. A **k-coloring** is a map $\phi : V(G) \to [k]$.

We need not use all k colors.

Definition 17.2. A **proper k-coloring** is a coloring with the constraint that $\phi(u) \neq \phi(v), \forall (u, v) \in E(G)$.

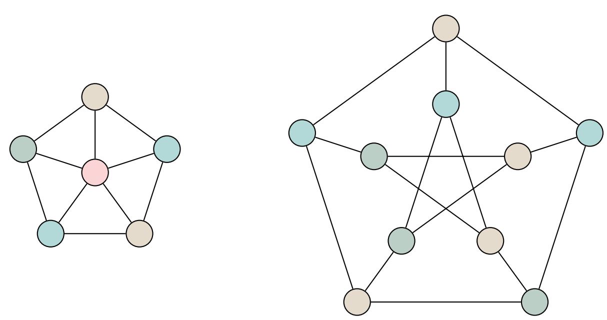
That is, we may not assign the same color to adjacent vertices.

Definition 17.3. A graph G is **k-colorable** if G has a proper k-coloring.

Definition 17.4. The **chromatic number** $\chi(G)$ is the minimum k such that G is k-colorable. A graph is **k-chromatic** if $\chi(G) = k$.

Let's look at some examples of k-chromatic graphs.

- For the complete graph K_n, we assign a different color to each vertex, so $\chi(K_n) = n$.
- Suppose S is an even cycle. Then $\chi(S) = 2$, with every other vertex the same color.
- Suppose S is an odd cycle. Then $\chi(S) = 3$. We cannot 2-color this graph since 2-colorable is equivalent to bipartite.
- Let G be the graph on the left. Then $\chi(G) = 4$ since the center must be a different color, and 3 colors are required for the 5-cycle.



- The Petersen graph has $\chi(G) = 3$.

Similar to matchings, colorings can represent problems. For example, a scheduling could be encoded in a graph, where edges correspond to conflicts. The solution corresponds to a proper coloring. In fact, any problem for which we have conflicts or constraints may be represented by a coloring problem.

Proposition 17.5

If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

The same coloring works for the subgraph.

Proposition 17.6

If G contains a k-clique, $\chi(G) \geq k$. So $\chi(G) \geq \omega(G)$ where $\omega(G)$ is the clique number (number of vertices in the largest clique).

This follows from proposition 17.5 and $\chi(K_n)$. Unfortunately, this bound is far from tight. In the next lecture, we will show that there are triangle-free graphs with arbitrary large chromatic number.

Proposition 17.7

A proper k-coloring is a partition of $V(G)$ into k independent sets, so

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

where $\alpha(G)$ is the independence number, or the number of vertices in the largest independent set.

We cannot do better than our largest independent set. However, it is not true that there must be a $\chi(G)$-coloring with $\alpha(G)$ vertices (we give a counterexample in the homework).

Proposition 17.8

For some graph G, suppose $V(G) = S \cup T$. Then $\chi(G) \leq \chi(G[S]) + \chi(G[T])$.

We color $G[S]$ with one set of colors, $G[T]$ with another set of colors, and combine the two colorings.

Proposition 17.9

Now suppose $G = G_1 \cup G_2$. Then $\chi(G) \leq \chi(G_1)\chi(G_2)$.

Suppose $\phi_1 : V(G_1) \to [k], \phi_2 : V(G_2) \to [\ell]$ are colorings for G_1, G_2. Then we can color G as $\phi : V(G) \to [k] \times [\ell]$, where we take a *pair* of colors for each vertex.

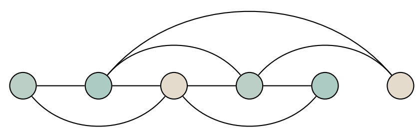
17.2 Greedy coloring

While it's nice to provide these loose bounds on colorings, they're not always that useful. To obtain more interesting properties of colorings, such as upper bounds on $\chi(G)$, we can use a strategy known as **greedy coloring**.

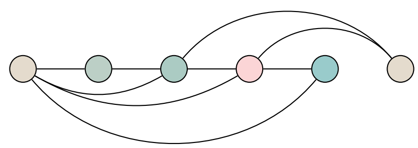
Example 17.10

Suppose a graph G has maximum degree at most d. Given an arbitrary ordering of the vertices, we color each vertex in turn. At each step, at most d colors are forbidden, so some color is always available. Thus G can be colored by $d + 1$ colors.

For general graphs, we may be clever about the ordering and reduce the number of colors needed. Consider the following graph.



If we color the vertices left to right, it is immediately apparent that we can color the graph with 3 colors. However, if we started at the third vertex, greedy coloring gives us 5 colors, which is not optimal.



The natural question to ask: when do we have a "nice" sorting that can be properly colored, greedily? This notion is related to the concept of degeneracy.

Definition 17.11. G is **k-degenerate** if every subgraph has minimum degree at most k.

Proposition 17.12

G is k-degenerate if and only if there exists an ordering $v_1, \ldots, v_n$ of $V(G)$ such that each v_i has at most k neighbors in $\{v_1, \ldots, v_{i-1}\}$.

Proof. We start with the $\Rightarrow$ direction. Since G is k-degenerate, G has a vertex of degree at most k. Fix this vertex as v_n (rightmost) and delete v_n. The remaining subgraph $G - v$ has a vertex with degree at most k. Set this as v_{n-1} and repeat.

Now we prove the $\Leftarrow$ direction. Given $v_1, \ldots, v_n$ and subgraph $H = (U, F)$ of G, take the largest index vertex v in U. We claim that the degree of v in H is at most k, since there are at most k edges going to the left in G. □

Theorem 17.13

If G is k-degenerate, then G is $k + 1$-colorable.

Proof. We do a greedy coloring via the ordering in proposition 17.12. □

Corollary 17.14

$\chi(G) \leq \Delta(G) + 1$, where Δ is the maximum degree.

Theorem 17.15 (Brooks' theorem)

If G is connected, then $\chi(G) \leq \Delta(G)$ unless G is a clique or an odd cycle, in which case $\chi(G) = \Delta(G) + 1$.

Proof. Proof provided in notes. □

17.3 Planar colorings

Last time we talked about planar graphs, so let's try to color them. There's the famous 4-color theorem, which states that every planar graph is 4-colorable. Unfortunately, ever proof involves computers bashing case work, but we can show some weaker results.

Lemma 17.16

Every planar graph has a vertex of degree at most 5.

Proof. Recall from last time that a planar graph of n vertices has at most $3n-6$ edges, so the average degree is at most $2e(G)/|G| < 6$. Thus there must be a vertex with degree at most 5. □

From lemma 17.16, we can deduce that planar graphs are 5-degenerate. We also emphasize that *every subgraph of a planar graph is also planar*. Thus, every planar graph is 6-colorable.

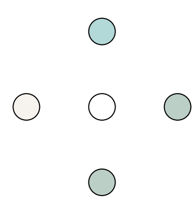
Remark 17.17. Suppose we want to find that $\chi(G) \leq k$. It is *not* sufficient to check that the minimum degree of G is at most k.

Graph coloring has been a favorite past time of mathematicians for centuries. At one point in the late 1800s, Kemp *thought* he found a proof for the 4-color theorem, and this proof stood for almost a decade until an error was found by Heawood. However, the latter salvaged Kemp's work by providing a proof for the 5-color theorem.

Theorem 17.18 (Heawood 1890)

Every planar graph is 5-colorable.

Let's take a detour and "try" to prove the 4-color theorem. Suppose v has degree 4. If $N(v)$ are colored with 3 colors, we may place the remaining color at v. Otherwise, if all 4 colors are used, we would like to "swap" a color.



TODO

This idea would work, except there is a planar graph with minimum degree 5—the icosahedron. However, the technique allows us to prove the 5-color theorem.

Proof of 5-color theorem. If $|V(G)| \leq 5$, then we are done, so we may assume $|V(G| > 5$. Let $v \in V(G)$ have degree ≤ 5. If the degree of v is < 5, we color $G - v$ and use a remaining color for v (we have 5 colors).

So assume that the degree of v is exactly 5. Fix a 5-coloring of $G - v$. Suppose the neighbors $v_1, \ldots, v_5$ of v are colored 1,2,3,4,5 in that order. If some color were missing, then we are done by coloring v using the missing color. Let G_{ij} be the subgraph of $G - v$ induced by vertices of colors i, j.

- If v_1, v_3 like in different components of G_{13}, then swap colors 1 and 3 in the component of v_1 in G_{13}. We free up color 1 for v.
- Assume there exists a path in G_{13} from v_1 to v_3. Likewise, there is a path in G_{24} connecting v_2 and v_4, but the latter path contradicts planarity. Therefore, we can swap a pair of colors in some component.

□

Theorem 17.19 (Appel-Haken 1977)

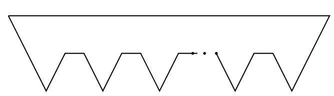
Every planar graph is 4-colorable.

All known proofs of this theorem use computers.

17.3.1 Art gallery problem

Example 17.20 (Art gallery problem)

We have a museum of some strange polygonal shape and we would like to place guards at some corners. How many guards are needed for an n-sided museum?



With m of these points, we need m guards, so here, there are $3m$ sides and m guards.

Theorem 17.21

$\lfloor n/3 \rfloor$ guards suffice.

First we triangulate the museum. We put vertices at corners and claim that this graph is 3-colorable.

Proof. We give a proof by induction. If we have 3 vertices, a triangle is 3-colorable. Otherwise, consider some edge (u, v), which splits the museum into two parts. We can 3-color each of the parts and label them so we can glue them together. □

Proof. We claim that G is 2-degenerate. If we consider only the inner edges, we have a tree (otherwise there is an inner cycle, and an inner wall, which is impossible). Then there are leaves, which correspond to corners that are not split in G. We remove that leaf and repeat. Eventually we find an ordering that satisfies the 2-degeneracy condition. Thus G is 3-colorable.

Now take a proper 3-coloring. At least one of the colors is used at most $\lfloor n/3 \rfloor$ times, so we put a guard at each vertex of this color.

This works because given any triangle, all 3 colors are present, so each guard watches over a triangle. □

18 November 29, 2018

In graph theory, we've discussed several topics, easy and hard. Here, we don't refer to how hard something is to understand, but rather, how "hard" it is to find a matching, or an Eulerian tour. For example, we cannot decide whether a given graph is Hamiltonian in polynomial time, but we can easily decide whether a graph has a Eulerian tour. Often, for many easy problems, there is the concept of duality.

- For König's theorem, we can find a vertex cover.
- For Menger's theorem, we have a separating set and the number of paths.

Generally, having some form of duality is a guarantee that your problem is "easy." Colorings are not easy. It is easy to check if a graph is 2-colorable—we just check if the graph is bipartite. However, it is NP-complete to check if a graph has chromatic number k, or simply, if it is 3-colorable.

18.1 Coloring, ctd

Today we will show some counter-intuitive proofs with colorings. For one, we can orient a graph—just assign directions to each edge so we have a directed graph.

Theorem 18.1

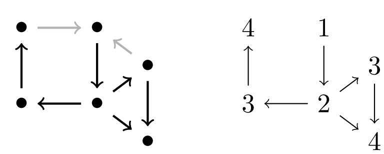
Let D be an orientation of graph G and let $\ell(D)$ be the length of the longest (directed) path in D. Then $\chi(G) \geq 1 + \ell(D)$. Furthermore, equality holds for some orientation. That is,

$$\chi(G) = \min_D \ell(D) + 1.$$

Proof. Let D' be a maximal acyclic subgraph of D (so D' is a directed acyclic graph, or a DAG). We give a coloring $f : V(G) \to [\ell(D) + 1]$, where

$$f(v) = 1 + \text{ length of longest path in } D' \text{ that ends in } v.$$

For example, the light edges are removed to form D', and the numbers on the right represent the colorings.



We claim that f is a proper coloring of G. Since D' is acyclic, f is strictly increasing along every path in D'. The key observation is that for $u, v \in D'$, where $f(u) < f(v)$ and u comes before v on a path, the longest path ending in u cannot contain v. Otherwise we have a cycle.

Now we finish checking that f is a proper coloring. If $(u, v) \in E(D')$, then we are done. Otherwise, if $(u, v) \in E(G) \setminus E(D')$, then $D' + (u, v)$ has a cycle containing (u, v) (by maximality of D'). Then there exists a path in D' from v to u, and $f(v) < f(u)$ since f increases along a path.

Now we prove the second part of this theorem. Take a proper coloring $f : V(G) \to [\chi(G)]$ of G. We orient vertices in order of ascending f. Then the length of the longest path cannot exceed $\chi(G) - 1$.

□

Last time, we also learned that $\chi(G) \geq \omega(G)$, where ω is the clique number. This bound can be very far from tight.

There exist G with $\omega(G) = 2$ (triangle-free) with arbitrarily large $\chi(G)$. This is known as Mycielski's construction.

1. Given graph G with $V(G) = v_1, \ldots, v_n$, we modify the graph by adding new vertices $u_1, \ldots, u_n$.
2. We add edges from u_i to all neighbors of v_i. We add edges from w to all the u_i.

If G is triangle-free, then G' is also triangle free, and $\chi(G') = \chi(G) + 1$.

Proof. First we check that G' is triangle-free. The original graph G had no triangles, so some new vertex must have formed a triangle. It is easy to check that, since all the u_i are connected to the neighbors of v_I, if G' has a triangle, then G has a triangle.

It is also easy to show that $\chi(G') \geq \chi(G)$, as the original coloring still works.

Finally, we check that if G' is $k+1$ colorable, then G is k colorable. Suppose w is colored by $k+1$. If some vertex v_i in $V(G) \subset V(G')$ were colored $k+1$, we can recolor v_i by the color of u_i. This is because u_i and v_i have the same neighbors in G. None of the u_i are colored by $k+1$ because w is colored by $k+1$, so we can create a coloring of G with k colors. □

With triangles, we can see the immediate neighborhood of each vertex—we are very myopic—and we cannot give an upper bound on the chromatic number. We might think that if we could see farther, we could eventually give an upper bound on the chromatic number. However, this is not true.

Theorem 18.2 (Erdös)

For every k, g, there exists a graph G with $\chi(G) \geq k$ and $\text{girth}(G) \geq g$.

Thus, chromatic number cannot be detected locally. Erdös proved this theorem by the probabilistic method.

18.2 Hadwiger's conjecture

Finally, we arrive at one of the most important open problems in graph theory. The 4-color theorem stated that all planar graphs are 4-colorable. Wagner's theorem also characterized that planar graphs avoid $K_{3,3}, K_5$ minors.

Conjecture 18.3 (Hadwiger's conjecture)**.** If $\chi(G) \geq t$, then G contains a K_t minor.

- This is trivial for $t = 1$ (every vertex must have a color).
- This is also trivial for $t = 2$ (we have an edge, which must be at least 2 colors).
- If $\chi(G) \geq 3$, G is not bipartite since it contains an odd cycle. Since it contains some cycle, we have a K_3 minor.
- We can show this is true for $t = 4$ with elementary operations (this could be a homework problem level hard).
- It turns out that $t = 5$ is equivalent to the 4-color theorem.

TODO write up better

We can explain the forward direction. If G is planar, then it is K_5 minor free, so by the contrapositive of Hadwiger's conjecture, $\chi(G) \leq 4$. The reverse direction also uses elementary operations.

Now we might ask, what happens for larger t? The 4-color theorem required massive computation for any proof, so does Hadwiger's conjecture also require massive case work? It turns out that for $t = 6$, the 4-color theorem suffices.

Robertson-Seymour showed that if G were a minimal count example, then there exists a vertex v such that $G - v$ is planar, so $\chi(G - v) \leq 4$, and thus $\chi(G) \leq 5$.

For $t \geq 7$, this is an open problem—one of the most notorious open problems in graph theory.

> "If in the future, you hear in the news that Hadwiger's conjecture has been proved, then that's big news."—yufeiz

Theorem 18.4 (Mader)

If the average degree of G is at least 2^{t-2}, then G contains a K_t minor.

Sketch of proof. We induct on $|V(G)|+t$. The base case is $t = 2$, which is trivial (this is an edge). Assume that we have a graph G with average degree $\geq 2^{t-2}$.

Case 1 Suppose we can find an edge $(u, v) \in E(G)$ where u, v have very few common neighbors. If we contract (u, v), we don't lose many edges, and hopefully we still have average degree $\geq 2^{t-2}$. By induction, we find a K_t minor in this contracted graph, which is a K_t minor in the original graph.

Case 2 Suppose there is no such edge. That is, all vertices u, v have many common neighbors. We pick some v and look at its neighborhood. $N(v)$ should have very large minimum degree since it has many common neighbors with neighbors $u \in N(v)$. By induction, we find a large K_{t-1} minor in this induced subgraph of $N(v)$. If we add v back in, we have a K_t minor.

□

Now we fill in the details.

Proof. **Case 1** If $\exists(u,v) \in E(G)$ such that u, v have less than 2^{t-3} common neighbors, then the average degree of $G \setminus \{u,v\}$ is

$$\frac{2e(G \setminus \{u,v\})}{n-1} \geq \frac{2(e(G \setminus \{u,v\}) - 2^{t-3})}{n-1}.$$

However, we assume that G has large average degree, so the above is equivalent to

$$\begin{aligned}\frac{2(e(G \setminus \{u,v\}) - 2^{t-3})}{n-1} &\geq \frac{2\left(2^{n-3}n - 2^{t-3}\right)}{n-1} \\ &= 2^{t-2}.\end{aligned}$$

We induct on this.

Case 2 $\forall(u,v) \in E(G)$, u, v have at least 2^{t-3} common neighbors.

Uh the math works out it's long.

□

The best result to date $ct\sqrt{\log t}$.

We do know the answer to many similar questions. These are all questions in extremal graph theory.

So far we've talked about vertex colorings, but we can also talk about edge colorings.

Definition 18.5. Given graph $G = (V, E)$, a **proper edge coloring** is a map $f : E(G) \to \mathbb{N}$ where no two incident edges share the same color.

We can convert a graph to its line graph $L(G)$ (edges become vertices and vice versa). An edge coloring of G is a vertex coloring of $L(G)$.

Definition 18.6. The **edge chromatic number** $\chi'(G)$ is the minimum number of colors required for a proper edge coloring of G.

It is clear that $\chi'(G) = \chi(L(G))$. For some easy bounds,

$$\begin{aligned}\Delta(G) \leq \chi'(G) &\leq 2\left(\Delta(G) - 1\right) + 1 \\ \omega(L(G)) \leq \chi(L(G)) &\leq \Delta(L(G)) + 1.\end{aligned}$$

Theorem 18.7 (Vizing's theorem)
$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$

This seems pretty easy—the edge chromatic number only has two options, right? Well sadly, deciding between them is NP-complete.

19 December 4, 2018

Here's a reminder that midterm 3 is on Thursday and this is the last lecture! :(In "time honored tradition," the professor brought us chocolate.

19.1 Ramsey theory

Ramsey theory begins with a puzzle.

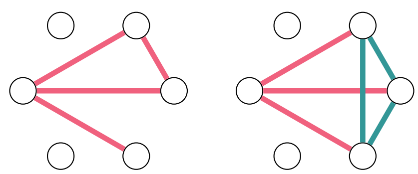
Example 19.1

Among 6 people in a room, there are 3 mutual acquaintances or 3 mutual non-acquaintances.

This statement is equivalent to the following:

> Every red-blue edge coloring of K_6 has either a red triangle or blue triangle.

Let's pick a vertex v. There are 5 edges going out of v, and by pigeonhole, there are three (red) edges incident to v. If there is any red edge between u_1, u_2, u_3, we have a red triangle. Otherwise, all edges between them are blue, and we have a blue triangle.



Ramsey theory deals with problems that find some "order" in large problems. Let's take a diversion on Ramsey. He lived a short life, dying at 28 due to liver problems, but he was very prolific during his few years. He wrote many influential papers, not only in mathematics, but also in philosophy and economics. Today we'll look at some foundational ideas in Ramsey theory.

Definition 19.2. The **Ramsey number** $R(s,t)$ is the smallest N such that every red-blue edge coloring of K_N contains either a red K_s or a blue K_t.

A priori, we do not know if this N is even well defined, but Ramsey showed that $R(s,t)$ is indeed finite.

- $R(3,3) = 6$, as we've seen.
- $R(2,t) = t, \forall t \geq 2$.

Theorem 19.3 (Erdös-Szekeres)

The next bound was

$$R(s,t) \leq \binom{s+t-2}{s-1}, \forall s,t \geq 2.$$

This bound is not believe to be tight, but it is some upper bound.

Proof. We will show that

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

and use induction to show that this recursion implies the desired inequality.

Consider a red-blue coloring of K_N, where $N = R(s-1,t)+R(s,t-1)$. Let's pick a vertex v. There are $N-1$ edges emanating from v, and by pigeonhole principle, there are at least $R(s-1,t)$ red edges or at least $R(s,t-1)$ blue edges incident to v.

Case 1 If there are lots of red edges, there exists at least a red K_{s-1} or a blue K_t. If there is a red K_{s-1}, we can add v to obtain K_s.

Case 2 If there are lots of blue edges, there exists at least a red K_s or a blue K_{t-1}, and we either add v for K_t or take the K_s.

This recursion tells us that $R(s,t)$ is always finite, so it proves Ramsey's theorem. To prove the final claim, we use induction on $s+t$.

We just showed that

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &= \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} \\ &= \binom{s+t-2}{s-1}. \end{aligned}$$

By Pascal's identity, the theorem holds. □

> "Let me give you a quick lesson on how to pronounce Hungarian names. There are lots of Hungarian names in combinatorics, partly due to Erdös's influence.
>
> *s* is pronounced as *sh*, and *sz* is pronounced as *s*."—yufeiz

Naturally, we might think about what happens with more colors. We can generalize $R(s,t)$. $R(s_1, s_2, \ldots, s_k)$ is the smallest N such that the edges of K_N, colored by $1, \ldots, k$, has a large monochromatic clique.

Theorem 19.4 (Ramsey's theorem for more colors)

xx

Proof. Fix a vertex v. By pigeonhole principle, v has lots of incident edges in some color. By repeating the argument for 2 colors,

$$\begin{aligned} R(s_1, s_2, \ldots, s_k) \leq & R(s_1 - 1, s_2, \ldots, s_k) - 1 \\ & + R(s_1, s_2 - 1, \ldots, s_k) - 1 \\ & \ldots \\ & + R(s_1, s_2, \ldots, s_k - 1) - 1 \\ & + 2. \end{aligned}$$

□

Alternate argument. Suppose we have 3 colors, a, b, c. We claim that

$$R(a, b, c) \leq R(R(a, b), c).$$

This reduces the problem for 3 colors into a problem for 2 colors.

Given a red-blue-orange coloring, we may temporarily treat red and blue as the same color. Then there exists a $R(a, b)$-clique of red-blue edges, or there exists a c-clique of orange edges. If the latter is true, we are done. Otherwise, by the definition of $R(a, b)$, either there exists an a-clique of red edges or a b-clique of blue edges. □

Definition 19.5. A **k-uniform hypergraph** has edges of k-tuples.

We use the notation $R^{(3)}(s, t)$ to denote the smallest N such that if we color all $\binom{N}{3}$ triples with red or blue, then there exists a red s-clique or a blue t-clique.

19.2 Lower bounds on Ramsey numbers

In the first half, we discussed upper bounds on Ramsey numbers. We know that

$$R(s, t) \leq \binom{s+t-2}{s-1}$$
$$R(s, s) \leq \binom{2s-2}{s-1} \leq 2^{2s} = 4^s.$$

Now we'll talk about some lower bounds.

Theorem 19.6 (Erdös 1947)
$\forall s \geq 3, R(s, s) > 2^{s/2}$.

The goal is to find a red-blue edge coloring without a monochromatic K_s. We give a proof by the probabilistic method (and the professor reminds us that he's teaching a graduate course on the probabilistic method next semester!)

Proof. We randomly color all edges red or blue and show that with positive probability, there is no monochromatic K_s.

For a fixed K_s subgraph, the probability that there is a monochromatic K_s is

$$2^{1-\binom{s}{2}}.$$

Given a random red-blue edge coloring of K_N, the probability that there *is* a monochromatic K_s (failure event) is

$$\leq 2^{1-\binom{s}{2}} \times \binom{N}{s}.$$

If we set $N \leq 2^{s/2}$, the failure probability is less than 1. So with positive probability, there is some coloring of K_N with no monochromatic K_s. Thus, $R(s, s) > N$. □

So far, we have shown that

$$\sqrt{2}^s < R(s,s) < 4^s.$$

There have been small improvements, but $\sqrt{2}$ and 4 have remained. There have been no exponential improvements since the 1940s, and this is a huge open problem in combinatorics.

Example 19.7 (Happy ending problem)
Every 5 distinct points in the plane have 4 points in convex position.

Proof. Look at the convex hull.

Case 1 5-gon, we are done.

Case 2 4-gon, we are done.

Case 3 3-gon, the two remaining points must be inside. The line connecting them intersects the triangle in two sides, so we can find the corresponding vertices.

□

This result is a Ramsey-theoretic statement. Can we always guarantee m points in convex position? As it turns out, yes.

Theorem 19.8 (Erdös-Szekeres)
$\forall m, \exists N$ such that for N distinct points in the plane with no 3 collinear, then exists an m-point subset in convex position.

How do we characterize points in convex position?

Claim 19.9. A finite set of points is in convex position if and only if every 4-point subset is in convex position.

Proof. The $\Rightarrow$ direction is trivial.

We prove the contrapositive of the $\Leftarrow$ direction. Suppose we have N points not in convex position. Consider their convex hull. There is at least one point v strictly on the interior of the convex hull.

If we triangulate the convex hull, v lies in one of the triangles (no 3 points collinear). Then the three points, given by that triangle, along with v, given 4 points not in convex position.

□

How is this at all related to graphs? TODO write better

1. Among 5 points, there exist 4 in convex position.
2. Claim 19.9

Let $N = R^{(4)}(m, 5)$. We color each 4-element subset red if these 4 points are in convex position, and blue otherwise. Ramsey's theorem tells us that either there is a red m-clique or a blue 5-clique. In the red clique, every 4 points are in convex position, so they are all in convex position (done). In the blue clique, no 4-point subset is in convex position, which is impossible by (1).

So what about bounds? For the longest time, it looked like $2^{m-2} < N < 4^m$. There was in fact, a conjecture that when $m = 4$,

3 years ago, $2^{m+\omega(m)}$.

Why is this the happy ending problem? Back in the 1940s, there were a bunch of Hungarian mathematicians who hung out together. Esther Klein came and inspired this duo. As a result of that interaction, Szekeres and Klein fell in love and both passed away in 2005, within an hour of each other, 70 years after they met.