Combinatorial Solution to the Staircase Problem and a Combinatorial Formula for the Fibonacci numbers.

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September 14, 2001

Abstract

We present a combinatorial solution to a well-known folklore problem, *the Staircase problem*. The problem has a conventional solution that uses the Fibonacci numbers $(1, 1, 2, 3, 5, 8, \ldots)$, and is a standard example used in presenting the Fibonacci numbers.

Our combinatorial solution to the Staircase problem yields an explicit formula for the Fibonacci numbers:

$$f_n = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-m-1}{m}, \text{ for } n \geq 1.$$

Unlike the well-known Binet's formula, this formula is expressed in terms of only rational quantities. The same formula appears in [1, p. 120,Eq. (2).] and [2].

1 The Staircase Problem

In this section we define the Staircase problem and then present two solutions for this problem: one is the standard recursive solution that yields the Fibonacci recursion, and the other is our combinatorial solution.

The Problem

Given a staircase with k stairs, in how many different ways can one climb the staircase if one can move either one or two steps forward at any stair?[1]

Let $s(k)$ denote the number of different ways to climb k stairs. A single stair can be climbed uniquely, by taking a single step; so, $s(1) = 1$. Two stairs can be climbed in two ways: One way is two take two single steps; the other is two take one double step. Hence, $s(2) = 2$. Figure 1 illustrates all five ways that one can climb a staircase that has four stairs. The staircase is drawn with straight lines; The arcs depict the steps taken.

[1] We use the word "stair" to refer to a single step of a stairway

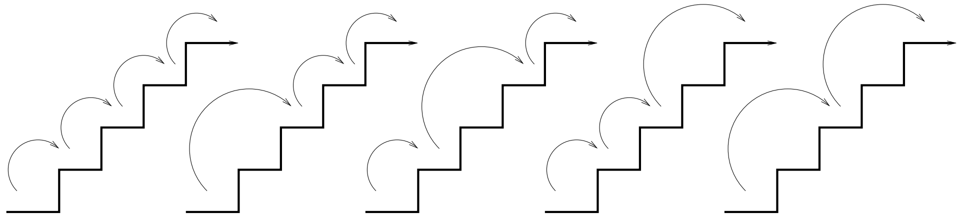

Figure 1: All different ways to climb four stairs using single and double steps.

Recursive solution

The standard solution to the Staircase problem is recursive: The very last step that reaches the kth stair can be made either by taking a single step from the $(k-1)$st stair or a double step from the $(k-2)$nd stair. Thus, the number of ways to climb k stairs is equal to the number of ways to climb $k-1$ stairs followed by a single step plus the number of ways to climb $k-2$ stairs followed by a double step. In other words, the formula for $s(k)$ can be written recursively as follows:

$$s(k) = \begin{cases} 1 & \text{for } \; k = 0 \text{ and } k = 1 \\ s(k-1) + s(k-2) & \text{for } k > 1. \end{cases} \tag{1}$$

The *Fibonacci numbers*, is a sequence "$1, 1, 2, 3, 5, 8, \ldots$", in which every term is equal to the sum of the two preceding terms. The standard definition of the Fibonacci numbers is recursive:

$$f_n = \begin{cases} 1 & \text{for } \; n = 1 \text{ and } n = 2 \\ f_{n-1} + f_{n-2} & \text{for } n \geq 3 \end{cases} \tag{2}$$

Compare equation (1) with this definition of the Fibonacci numbers. They are virtually the same. The number of ways to climb k stairs is the same as the $(k+1)$st Fibonacci number, or in other words:

$$f_n = s(n-1). \tag{3}$$

Equation (2) is recursive; the value of f_n is expressed in terms of the values of f_{n-1} and f_{n-2}. There is also an explicit formula, known as the *Binet's Formula*, for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n. \tag{4}$$

Combinatorial solution

The staircase problem can also be solved combinatorially as the problem of arranging single and double steps on k stairs.

Denote by $r(k, m)$ the number of different ways to climb k stairs so that the climb includes exactly m double step. The total number, $s(k)$, of ways one can climb the staircase can then be expressed as the following sum:

$$s(k) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} r(k,m). \tag{5}$$

That is, the number of ways to climb a staircase is the the number of ways to climb with no double steps, plus the number of ways to climb with one double step, plus the number of ways to climb with two double step, etc.

But what is $r(k,m)$? Notice that, when a climb of k stairs includes m double steps, the total number of steps taken is $k-m$; this is because there are m double steps, and $k-2m$ single steps. Hence, $r(k,m) = \binom{k-m}{m}$, and the Staircase problem has the following solution:

$$s(k) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-m}{m}. \tag{6}$$

By combining equations (6) and (3), we derive the following explicit formula for the nth Fibonacci number.

$$f_n = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-m-1}{m}, \text{ for } n \geq 1 \tag{7}$$

Notice that unlike the Binet's formula, this combinatorial formula contains only rational quantities. After we "discovered" this formula, we found that it appears in [1, p. 120,Eq. (2).] and [2].

To illustrate equation (7), consider the case of $n = 6$:

$$\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8,$$

which is indeed the sixth Fibonacci number.

2 Algebraic Derivation

In this section we verify equation (7) by showing that $f_n = f_{n-1} + f_{n-2}$. The derivation relies on the following property of Binomial coefficients:

$$\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}, \tag{8}$$

for any integer j and any nonnegative integer i, where $\binom{i}{j} = 0$ for $j < 0$ and for $j > i$ [3, Sec. 4.5, p. 312].

Consider equation (7).

$$f_n = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-m-1}{m}$$

By using equation 8, we get

$$= \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-m-1-1}{m} + \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{(n-m-1-1}{m-1}$$

Now, in the first sum, consider separately the cases of n being even and odd. For the latter case use the fact that $\binom{i}{j} = 0$ for $j > i$. In the second sum, do the substitution $l = m - 1$ and use the fact that $\binom{i}{j} = 0$ for $j < 0$. After these transformations, we get:

$$\begin{aligned} &= \sum_{m=0}^{\lfloor \frac{(n-1)-1}{2} \rfloor} \binom{(n-1)-m-1}{m} + \sum_{l=0}^{\lfloor \frac{(n-2)-1}{2} \rfloor} \binom{(n-2)-l-1}{l}, \\ &= f_{n-1} + f_{n-2}. \end{aligned}$$

3 Concluding Thoughts

Impact on educational programs

Fibonacci numbers occupy an established, permanent place within Mathematics and Computer Science, and in particular, within the educational settings of these fields. Fibonacci numbers are used to present such notions as recursive definitions and recurrent formulas.

Fundamental combinatorial counting techniques are often taught within the same courses in which recursions and recurrences are taught.

General public seems to be unaware of the existence of the combinatorial formula for Fibonacci numbers. We stumbled upon this formula by devising a combinatorial solution to a problem that has a standard recursive solution using the Fibonacci numbers. It is only after an extensive literature search that we have discovered it published in [1] and [2].

Problems like the Step Counting problem are used to illustrate the Fibonacci numbers. They can also be used to illustrate combinatorial counting techniques and to provide connection between Combinatorial Counting and Recurrences.

Impact on applications

Fibonacci numbers have a number of real world applications (see numerous books on the subject). It is interesting to see whether the combinatorial formula for the Fibonacci numbers can help explain this connection.

References

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