Today.

Undirected S-T Conn revisited (deterministic log space)
Undirected $s-t$ connectivity revisited

given: under $G$
node $s,t$

question: are $s,t$ in same component?

an easy case:

**def:** $(N,0,\lambda)$-graph

- nodes
- degree
- upper bound on $\lambda$ of transition matrix

**a well-known fact:** Tanner, Abh-Halman

$\lambda < 1$, $\exists \varepsilon > 0$ such that $(N,0,\lambda)$-graphs $G$

+ for $A \subseteq V$, $|A| < N/\alpha$ implies $|N(A)| \geq (1+\varepsilon)|A|$

i.e. $G$ "expands"

fact implies another easy fact: such a $G$ also has $O(\log N)$ diameter

Idea for algorithm in which each component of graph is $(N,0,\lambda)$ for $\lambda = O(\log N)$ or just $\log n$-diameter

- enumerate all $D^e$ paths (for $e = O(\log N)$) starting at $s$
- if ever see $t$, output "connected"
Space requirements:
assume lexicographic ordering on paths (comes from ordering on outedges)
just keep track of DFS path from s:
- const # bits per step of path
- $O(\log n)$ length
Total: $O(\log n)$ bits

Behavior:
if s,t not connected, never answers connected
if s,t connected - will find path

Problem: not all graphs are $(N,0,\lambda)$-graphs for $\lambda < 1$
or even just $O(\log n)$ diameter
or even just constant degree

In general, we know:
Theorem: connected, non-bipartite graphs, $\lambda(G) \leq 1 - \frac{1}{DN^2}$
not too good!
What about powering?

if \( G \) is \((N, D, \lambda)\) then \( G^4 \) is \((N, D^4, \lambda^4)\)

good or bad?

+ reduce 2nd e-val
+ increase degree

Need operation which reduces degree w/o killing 2nd e-val

i.e. 1) if it was expander, should remain so
   but reduce degree

2) don't need to increase expansion, powering
   will do that

Let's start with a "base graph"

Thm 1: const \( D_e \) + \((D_e^{16}, D_e, \frac{1}{D_e})\)-graph

\[ \uparrow \quad \uparrow \quad \uparrow \]

\[ N \quad O \quad \lambda \]

constant size - can find this by enumeration
First issue to handle:

nice to have regular graph of const degree with same connected components

One way to transform $G$:

- Quadratic blowup in # of nodes, but just once
- Can add self-loops to deg 0 nodes

in both cases, easy to fix neighbor find in log space

Could be bad for $O$, but we'll fix later...

Second issue: representing graphs

- Adjacency lists?
- Rotation map $\text{Rot}_G : [N] \times [N] \rightarrow [N] \times [N]$

$\text{Rot}_G(v, i) = (w, j)$ if $i$th edge of $v$ leads to $j$th edge of $w$ leads to

allows back and forth on same edge!
1. Replacement Product $G \circ H$

Given $G$, $d$-regular, $N$ nodes $G'$ with $N: D$ nodes degree $d+1 > c$

- nodes: $v \in G$ replaced by copy of $H$
- edges: each vertex $H_v$ connected to nbrs in $H_v$
  - if $u$ is $i$th nbr of $v$ in $G + V$ is $u$'s $j$th nbr
  - add edge from $i$th node of $H_v$ to $j$th node of $H_u$

2. Zig-zag Product $G \otimes H$

Given $G$, $d$-regular, $N$ nodes $G''$ with $N: D$ nodes degree $d^2$

- nodes: as in $G'$, $v \in G$ replaced by copy of $H$
- edges: path of length 3 in $G''$, i.e., $(u,v) \in E''$ iff $u \in H_i$, $(u,w) \in E(H_i)$, $(w,z) \in E(G)$, $(z,v) \in E(H_j)$
  - $d$ choices (local)
  - $1$ choice (between clouds)
  - $d$ choices (local)

"clouds" - each is a copy of $H$
Example

\[ G \quad H \quad \Rightarrow \quad G \circ H \]

\[ (\text{Assume self loops}) \]
Some Inuition:

In terms of cuts:

- to find min cut, would want to break s.t. clouds intact (since clouds are expanders)
  \[ \Rightarrow \text{cut size should be similar to } G's \]

In terms of random walks:

Two extreme cases:

1) distribution far from uniform in each cloud:
   then walks on it make it more uniform
   \[ \text{and step will affect} \]

2) uniform within clouds but different weights on clouds:
   then walks on it won't affect,
   \[ \text{and walking on } G \text{ improves slowly} \]
For \( \alpha \leq \frac{1}{\sqrt{2}} \)

\[ \text{Thm 1:} \quad \text{for } G \text{ an } (N, 0, \lambda)-\text{graph + } H \text{ a } (D, d, \alpha)-\text{graph, } G \otimes H \text{ is } (N D, D^2, \alpha^2). \]

\[ \text{st.: } \frac{1}{2} (1 - \alpha^2) (1 - \lambda) \leq 1 - \lambda \]

So,

\[ \lambda \leq 1 - \frac{1}{2} (1 - \alpha^2) (1 - \lambda) \]

\[ \leq 1 - \frac{3}{8} (1 - \lambda) \]

\[ \leq 1 - \frac{1}{3} (1 - \lambda) \]

\[ \leq \frac{2}{3} + \frac{1}{3} \lambda \quad \text{still } < 1, \text{ now, let's power it up a few times!} \]

How do we use this?

Main Transformation:

Given: \( G \) \( D^{16} \)-regular on \( N \) nodes

\( H \) \( D \)-regular on \( D^{16} \) nodes \( \quad (\text{Thm 1}) \)

Transformation:

Let smallest int st.: \( (1 - \frac{1}{2N^2})^x \leq \frac{1}{2} \)

\( G_0 \leftarrow G \)

\( G_x \leftarrow (G_{x-1} \otimes H)^8 \) \quad \text{powering}

Output: \( G_x \) \quad \text{degree reduction}
What are properties of $G_e$?

$\# \text{ nodes} = N \cdot (D^{16})^e$ which is $\operatorname{poly}(N)$ since

$$l = O(\log N)$$
$$+ a = O(1)$$

Lemma if $\lambda(H) \leq \frac{1}{2} + \frac{1}{6}$ connected, not bipartite

then $\lambda(G_e) \leq \frac{1}{3}$

Proof idea

$$\lambda_{G_0} \leq 1 - \frac{1}{DN^2}$$

Need Claim

$$\lambda_{G_e} \leq \max \left\{ \lambda(G_0)^2, \frac{1}{2} \right\}$$

If Claim, then for $d = \Theta(\log PN)$ have

$$\lambda(G_e) \leq \max \left\{ \lambda(G_0)^2, \frac{1}{2} \right\} \leq \frac{1}{2}$$

Then prove claim by induction on $i$. $\leq \frac{1}{2}$
Final Algorithm

1. preprocess $G$ to make regular, nonbipartite
   with same connected components
   (can do by $G \circ N$-cycle then add self-loops)
   new graph has $N^4$ nodes — or can use idea on pg. 4
   + self-loops

2. use zigzag + powering transformation to get $G_e$

3. run expander algorithm on $G_e$

A final issue:
how do you perform walks in logspace?

need to show that

can compute
rotation map
of $G$ given
rot map of $G, H$

use rotation maps!
give way of going backwards forwards on
a path

⇒ only need to remember a constant
# of bits to choose next step of
path