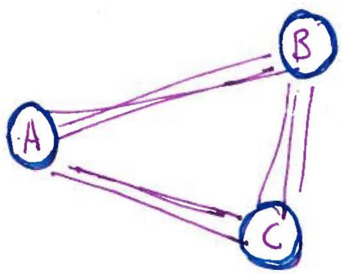
Today:

- Szemerédi's Regularity Lemma
- Testing dense graph property of Δ-freeness

Graphs with "random" properties:

Example Question:

How many triangles in a random tripartite graph?



density η

$\forall\ u \in A,\ v \in B,\ w \in C:$

$$\Pr[u \sim v \sim w] = \eta^3 \qquad \delta_{u,v,w} = \begin{cases} 1 & \text{if } u \sim v \sim w \\ 0 & \text{o.w.} \end{cases}$$

$$E[\delta_{u,v,w}] = \eta^3$$

$$E[\#\text{ triangles}] = E\Big[\sum_{\substack{u \in A \\ v \in B \\ w \in C}} \delta_{u,v,w}\Big]$$

$$= \eta^3 |A|\,|B|\,|C| \qquad \blacksquare$$

What weaker assumptions can we make to get similar bounds?

One possibility:
Density + regularity of set pairs:

<u>def</u> For $A, B \subseteq V$ s.t.

(1) $A \cap B = \varphi$

(2) $|A|, |B| > 1$

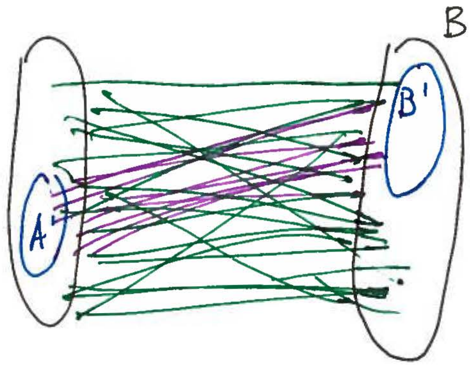
Let $e(A,B)$ = # edges bet A + B

+ density $d(A,B) = \dfrac{e(A,B)}{|A|\,|B|}$

Say A,B is γ-regular if $\forall\ A' \subseteq A,\ B' \subseteq B$
s.t. $|A'| \geq \gamma |A|$
$|B'| \geq \gamma |B|$

behaves like a "random" graph

$$|d(A',B') - d(A,B)| < \gamma$$



So lose only factor of 16

Lemma [Komlos Simonovits]
(density)

$\forall \eta > 0$ $\exists\ \gamma$ (regularity parameter, depends only on η) $= \frac{1}{2}\eta = \gamma^{\Delta}(\eta)$

δ (# triangles, depends only on η) $= (1-\eta)\dfrac{\eta^3}{8} \geq \dfrac{\eta^3}{16} = \delta^{\Delta}(\eta)$

if $\eta < \frac{1}{2}$

s.t. if A,B,C disjoint subsets of V s.t. each pair is γ-regular with density $> \eta$ then G contains $\geq \delta \cdot |A|\cdot|B|\cdot|C|$ distinct Δ's with vertex from each of A,B,C.

Proof ([Alon Fischer Krivelevich Szegedy] simplify).

$A^* \leftarrow$ nodes in A with $\geq (\eta-\gamma)|B|$ nbrs in B
$\geq (\eta-\gamma)|C|$ nbrs in C

Claim $|A^*| \geq (1-2\gamma)|A|$

Pf of Claim

$A' \leftarrow$ "bad" nodes of A wrt. B (i.e. $< (\eta-\gamma)|B|$ nbrs in B)
$A'' \leftarrow$ " " " " " C (" " " " " C)

then $|A'| \leq \gamma|A|$ (and $|A''| \leq \gamma|A|$)

why? otherwise: look at pair A', B ← size $|B| \geq \gamma|B|$

size $\geq \gamma|A|$ by assumption

by def of A'

$$d(A', B) < \frac{|A'|\cdot(\eta-\gamma)|B|}{|A'|\,|B|} = \eta-\gamma$$

but $d(A, B) \geq \eta$
so $|d(A', B) - d(A, B)| > \gamma$

$\rightarrow\leftarrow$ to γ-regularity

(simalarly for $|A''| \leq \gamma|A|$)

But $A^* = A \setminus (A' \cup A'')$

so $|A^*| \geq |A| - |A'| - |A''|$
$\geq |A| - 2\gamma|A| = (1-2\gamma)|A|$

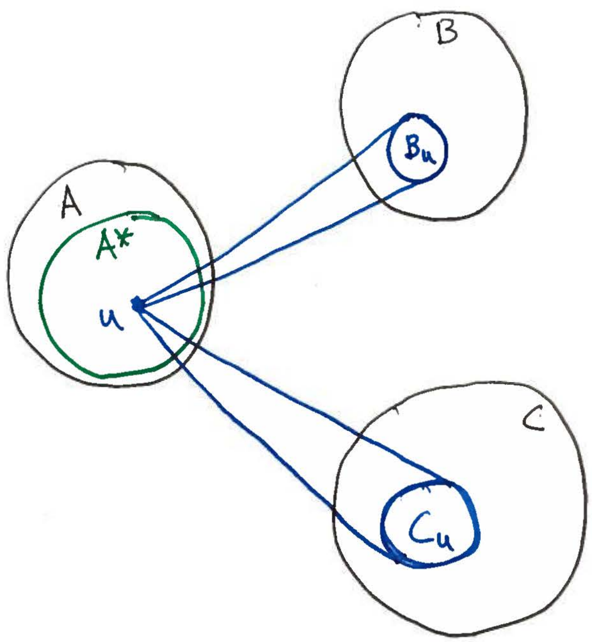
■ end-proof-of-claim

Finishing proof of Lemma:

For each $u \in A^*$:

def. $B_u \equiv$ nbrs of u in B so $|B_u| \geq (\eta-\gamma)|B| \geq \gamma|B|$

$C_u \equiv$ nbrs of u in C so $|C_u| \geq (\eta-\gamma)|C| \geq \gamma|C|$



Since γ chosen s.t. $\gamma < \frac{\eta}{2}$, we have $\eta - \gamma > \gamma$

Note: # edges between $B_u + C_u \Rightarrow$ lower bound on # distinct triangles with u as a vertex

$d(B,C) \geq \eta$

$\Rightarrow d(B_u, C_u) \geq \eta - \gamma$ (since $|B_u|, |C_u|$ big enough + B, C γ-regular)

$\Rightarrow e(B_u, C_u) \geq (\eta-\gamma)|B_u||C_u|$

$\geq (\eta-\gamma)^3 |B||C| \equiv$ l.b. on # triangles with u

$\Rightarrow$ total # Δ's $\geq (1-2\gamma)|A| \cdot (\eta-\gamma)^3 |B||C|$

$\geq (1-\eta)(\eta/2)^3 \cdot |A|\cdot|B|\cdot|C| = (1-\eta)\frac{\eta^3}{8} \cdot |A|\cdot|B|\cdot|C|$

Choosing $\gamma = \eta/2$

∎

Do any interesting graphs have regularity properties?

in some sense, all graphs do! i.e. every graph (in some sense) can be approximated by random graphs.

Szemerédi's Regularity Lemma

would like it to say:

"one can equipartition the nodes V into $V_1 \dots V_k$ (for some const k) s.t. all pairs (V_i, V_j) are ε-regular"

only most i.e. $\leq \varepsilon \binom{k}{2}$ don't have to be regular

k is const > 1 sometimes need $k > m$ for some m ($k=1$, $k=n$ trivial)

A more useful version:

Lemma

$\forall\ m, \varepsilon > 0 \quad \exists\ T = T(m, \varepsilon)$ s.t. (huge constant)

given $G = (V, E)$ s.t. $|V| > T$

+ A an equipartition of V into sets

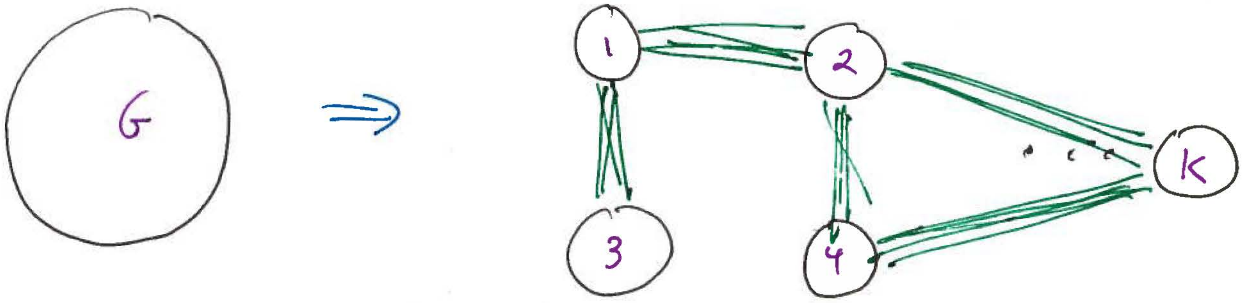
then $\exists$ equipartition β into k sets which refines A,

s.t. $m \leq k \leq T$

+ $\leq \varepsilon \binom{k}{2}$ set pairs not ε-regular

Note: T does not depend on $|V|$

"Picture":



Why is this good?

- partition big graph into "constant" # partitions st. each pair behaves like random bipartite graph
- random bipartite graphs have nice properties

Why was SRL first studied?

to prove conjecture of Erdös & Turán

sequences of ints must always contain long arithmetic progressions.

Very rough idea of a proof:

$$\text{ind}(V_1 \cdots V_k) = \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} d^2(V_i, V_j) \leq 1/2$$

if a partition violates, can refine s.t.

ind $(V_1' \cdots V_{k'}')$ grows significantly (i.e. by $\approx \varepsilon^c$)

so in less than $\frac{1}{\varepsilon^c}$ refinements,

have good partition

How many classes (k')?

u.b.: tower of size $\frac{1}{\varepsilon^c}$: $\underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}}_{\frac{1}{\varepsilon^c}}$

l.b.: " " " $\frac{1}{\varepsilon^{c'}}$