Today:

Testing $\Delta$-freeness
An application of the SRL:

Property testing of a graph: is it triangle-free?

Given: graph $G$, adjacency matrix format

Desired Behavior:
- If $G$ is $\Delta$-free, output Pass
- If $G$ is $\varepsilon$-far from $\Delta$-free, then $\Pr[\text{output FAIL}] > \frac{1}{2}$

must delete
$\geq 2\varepsilon n^2$ edges

make it $\Delta$-free

How much time does this require?

trivial $O(n^3)$, $O(n^2)$, $O(n)$, $O(1)$?

Algorithm

Do $O(\varepsilon^{-1})$ times

Pick $V_1, V_2, V_3$

if $\Delta$ reject & halt

Accept

← constant time
*Theorem*  \( \exists \epsilon, \delta \in \mathbb{R}^+ \) s.t. \( A \subseteq G \) s.t. \( |V| = n \)

+ s.t. \( G \) is \( \epsilon \)-far from \( \Delta \)-free,
+ then \( G \) has \( \geq \delta(n^{3/2}) \) distinct \( \Delta \)'s.

**Concealed Algorithm** has desired behavior

**Proof of Corr (Given Thm)**

- if \( \Delta \)-free, accepts \( \checkmark \)
- if \( \epsilon \)-far,

\[ \equiv \delta(n^{3/2}) \Delta \]'s \]

- each loop passes with prob \( \leq 1 - \delta \)

\[ \Pr \text{ I don't find } \Delta J \leq (1 - \delta)^{c/8} \]

\[ \leq e^{-c} < \frac{1}{3} \]

**Proof of Thm**

Use regularity to get equipartition \( V_1, \ldots, V_k \)

s.t. \( \frac{\epsilon n}{k} \leq k \leq T(\delta \epsilon^{-1}, \epsilon') \)

equivalently: \( \frac{\epsilon n}{k} \geq \frac{n}{k} \geq \frac{n}{T(\delta \epsilon^{-1}, \epsilon')} \)

(do this by starting with arbitrary equipartition into \( s/8 \) sets as \( A \))

for \( \epsilon' \equiv \min \left\{ \frac{\epsilon}{3}, \sqrt[3]{\epsilon} \right\} \)

s.t. \( \leq \epsilon'(\frac{k}{n}) \) pairs not \( \epsilon' \)-regular
Need: \# of partitions fairly large st. \# edges inside a partition not too big

Assume \( n/k \) is integer

\( G' \equiv \text{take } G \text{ and }

1) delete edges of \( G \) internal to any \( V_i \)
   how many?
   \[
   \leq \frac{n}{k} \cdot n \leq \frac{\varepsilon n^2}{5}
   \]
   \[
   \text{deg w/in } V_i \quad \text{sum over all nodes}
   \]

2) delete edges between \( \varepsilon' \)-non-regular pairs
   how many?
   \[
   \leq \varepsilon' \left( \frac{n}{k} \right)^2 \leq \frac{\varepsilon}{5} \cdot \frac{n^2}{k} \leq \frac{\varepsilon \cdot n^2}{10}
   \]
   \[
   \text{Max \# edges per pair - here we use equipartition } \Rightarrow \text{max size of } V_i \text{ is } \approx \frac{n}{k} \left( 1 + \varepsilon \right)
   \]

3) delete edges between low density pairs
   how many?
   \[
   \leq \frac{\varepsilon}{5}
   \]
   \[
   \leq \sum_{\text{low density}} \left( \frac{n}{k} \right)^2 \text{ note } \sum \left( \frac{n}{k} \right)^2 \leq \left( \frac{n}{k} \right)^2 \leq \frac{\varepsilon n^2}{10}
   \]

Total deleted edges from \( G' \): \( \leq 3n^2 \)
But, \( G \) was \( \epsilon \)-far from \( \Delta \)-tree
so \( G' \) must still have a \( \Delta \)!!!

(the \( \Delta \) must be 1) in 3 distinct \( V_i, V_j, V_k \) since
removed inter partition edges
2) regular pairs - since removed non-regular
3) high density pairs - since removed low density pairs
by construction of \( G' \))

\[
\forall i,j,k \text{ distinct s.t. } x \in V_i, y \in V_j, z \in V_k
\]
\[
V_i, V_j, V_k \text{ all } \approx \frac{\eta}{5} \text{ density pairs } \quad \Leftarrow \text{ not just a } \Delta, \text{ but a high density } \Delta!!!
\]

\[\Delta\text{-counting lemma } \Rightarrow \]
\[
\geq s^2 \left( \frac{\epsilon}{8} \right) \left| V_i \right| \left| V_j \right| \left| V_k \right| \text{ triangles in } G'
\]
where \[s^2 = \left( 1 - \eta \right) \frac{\eta^3}{8} \]

\[
\geq \frac{s^2 \left( \frac{\epsilon}{8} \right) \eta^3}{(T(5\epsilon', \epsilon'))^3} \Delta's
\]

\[
\geq \left( \frac{\eta}{5} \right) \Delta's \text{ in } G'
\]
(and thus in \( G \))

\[
\geq \left( \frac{\eta}{5} \right) \Delta's \text{ in } G'
\]

for \( \delta = 6 \delta \left( \frac{\epsilon}{8} \right) \left( T(5\epsilon', \epsilon') \right)^3 \)

\[
\geq \frac{1}{16000} \frac{\epsilon^3}{8} = \frac{\epsilon^3}{6400}
\]
Extensions

- Koulo-Simnovits holds for all constant sized subgraphs

- almost "as is" can use method to test all 1st-order graph properties

\[ F_{u_1 u_2 u_3 \ldots u_k} \forall v_1 \ldots v_2 R(u_1 \ldots u_k v_1 \ldots v_2) \]

\[ \text{defined by } v, \lambda, T \text{ in } \text{ners} \]

i.e.

\[ \forall u_1 u_2 u_3 R(u_1 u_2 u_3) \]

\[ \text{encodes } \lambda(u_1 u_2, u_2 u_3, u_1 u_3) \]

H-freeness for constant size H

Induced \[ \square \]

not induced \[ \Box \]

Forbidden \[ \square \]

H-sided constant time \( \prec \) hereditary graph props [Alon Shapira]

closed under vertex removal (not necessarily edges)

includes monotone graph props

chordal

perfect

interval graph

difficulty: infinite set of forbidden subgraphs also forbidden as induced.

H-sided constant time \( \prec \) regular partition is hardest testing problem

properly testable iff can reduce to testing [Alon Fischer, Norman Shapira]

if satisfies one of finitely many Seymour's partitions.