Today:

- Derandomization via method of Conditional Expectations
- Random Walks!

Markov chains + random walks on graphs

Stationary Distributions
More derandomization: The method of conditional expectations

Idea: view coin tosses of algorithm as path down a tree of depth \( m \leq \# \text{ coin tosses} \)

\[ \delta = H, \quad l = T \]

\[ \text{good} = \text{correct/randomized Pass... good bad bad bad good good} \]

good randomized algorithm \( \iff \) most leaves are good

more formally:

Fix randomized algorithm \( A \)
input \( x \)
\( m = \# \text{ random bits used by } A \text{ on } x \)

for \( 1 \leq i \leq m \) \( + \ r_i \text{...} r_i \in \{0,1\} \), let \( p(r_1...r_m) = \text{ fraction of continuations that end in "good" leaf} \)

\[
p(r_1...r_m) = \frac{1}{2} \cdot p(r_1...r_m, 0) + \frac{1}{2} \cdot p(r_1...r_m, 1)
\]

by averaging, if setting of \( r_i \) to 0 or 1

\[ p(r_1...r_m, 0) \geq p(r_1...r_m, 1) \] Can we figure this out?
if \( p(r_1, \ldots, r_n) \geq p(r_1, \ldots, r_k) \) \( \forall k \)

then \( p(r_1, \ldots, r_m) \geq p(r_1, \ldots, r_{m-1}) \geq \cdots \geq p(r_1) = p(r) \geq \frac{1}{3} \)

\[\uparrow\]

this is a leaf
so value is 1 or 0,
but if \( \geq \frac{1}{3} \)
must be 1

fraction of good paths

main issue: how do we figure out the best setting of \( r_{\text{th}} \) at each step?

An example: Max Cut (another way to derandomize)

Recall algorithm:

- Flip \( n \) coins \( r_1, \ldots, r_n \)
- Put node \( i \) in \( S \) if \( r_i = 0 \) \( \oplus T \) if \( r_i = 1 \)
- Output \( S, T \)

derandomization:

\[ e(r_1, \ldots, r_n) = E_{R_{\text{th}} \ldots R_N} \left[ | \text{cut}(S, T) | \right] \text{ given } r_1, \ldots, r_n \] choices made

\[ e(\Lambda) \geq \frac{|E|}{2} \] (from previous lecture)

how do we calculate \( e(r_1, \ldots, r_n) \)?
Let

\[ S_{i+1} = \{ j \mid j \leq i+1, \ r_j = 0 \} \]

\[ T_{i+1} = \{ j \mid j \leq i+1, \ r_j = 1 \} \]

\[ V_{i+1} = \{ j \mid j \geq i+2, \ r_j \leq 3 \} \]

So

\[ e(r_1 \ldots r_{i+1}) = (\# \text{ edges between } S_{i+1} + T_{i+1}) + \frac{1}{2}(\# \text{ edges touching } V_{i+1}) \]

Note: don’t need to calculate \( e(r_1 \ldots r_{i+1}) \)

just need to compare \( e(r_1 \ldots r_i, 0) \) vs. \( e(r_1 \ldots r_i, 1) \) - is it \( \geq 3 \)

Note:

- \( V_{i+1} \) term is same for both
- first term differs only on edges adjacent to node \( i+1 \)

\[ \text{to maximize this, place node } i+1 \]

\[ \text{to maximize cut size} \]

\[ \text{i.e. } \left| \text{# edges between node } i+1 + S_i \right| \]

\[ \text{vs. } \left| \left( \ldots \right) + T_i \right| \]
Corresponds to:

Greedy Algorithm:

1) \( S \leftarrow \emptyset, \ T \leftarrow \emptyset \)

2) For \( i=0 \ldots N-1 \)
   
   place node \( i \) in \( S \) if \( \# \text{edges between } i + T \geq \# \text{edges into } i + S \)

   else place in \( T \)
Random walks

Markov chains:

$\Omega = \text{set of "states" (or nodes)}$ (here always finite)

$x_0, \ldots, x_t \in \Omega$ sequence of visited states

Markovian property:

$$\Pr[X_{t+1} = y \mid X_0 = x_0, X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t]$$

$$= \Pr[X_{t+1} = y \mid X_t = x_t]$$

Next step depends only on where you are, not how you got there.

Wlog, assume transitions independent of time:

i.e. $P(x,y) = \Pr[X_{t+1} = y \mid X_t = x]$.

so can use "transition matrix" to represent it

Important special case:

transitions uniform on subset corresponding to neighbors of node

$\text{def. random walk on } G = (\Omega, E)$

is a sequence $S_0, S_1, \ldots$ of nodes

where $S_0$ is a start node.

At each step $i$, $S_{i+1}$ picked uniformly from $N(S_i)$ outdegrees.
Markov chain

\[ p(i,j) = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2}
\end{bmatrix} \]

random walk on digraph

\[ p'(i,j) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix} \]

\[ d(i) = \text{# out edges of node } i \]

\[ p(i,j) = \begin{cases} 
\frac{1}{d(i)} & \text{if } (i,j) \in E \\
0 & \text{o.w.}
\end{cases} \]

\[ \forall i, \sum_j p(i,j) = 1 \]
Distributions after $t$ steps

Transition probabilities for $t$ steps: $P^t(xy) = \begin{cases} P(xy) & t=1 \\ \sum \frac{P(xz)}{2} P^t(z,y) & t > 1 \end{cases}$

Initial distribution: $\Pi^0 = (\Pi_1^0, ..., \Pi_n^0)$ where $\Pi_i^0 = \Pr[\text{start at node } i]$

Distribution after one step:

$\Pi^1 = \Pi^0 \cdot P = (\sum \frac{P(z,1)}{2} \Pi(z), \sum \frac{P(z,2)}{2} \Pi(z), ...)$

$t$-step distribution: $\Pi^t = \Pi^0 \cdot P^t$

Finite Markov Chain Properties

Stochastic matrix: rows of $P$ sum to 1.

doubly stochastic matrix: rows and columns sum to 1.

e.g. random walk on undirected graph or digraph in which indegree = outdegree = const for all nodes.

irreducible: ("strongly connected") $\forall x,y \exists t = c(xy) \text{ s.t. } P^t(xy) > 0$

ergodic: change of quantifier order $\exists t_0 \text{ s.t. } \forall t > t_0 \forall x,y \ P^t(xy) > 0$ (stronger than irreducible, why?)
Aperiodic:\n\forall x \quad \gcd \exists t : p^t(x, x) > 0 \Rightarrow \gcd = 1 \not\Rightarrow \text{not bipartite, k-partite...}

Thm \quad \text{Ergodic } \iff \text{Irreducible + Aperiodic}

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**Stationary Distributions**

Does it depend on $\Pi_0$? \quad \exists \text{stationary distribution } \Pi \quad \forall y \quad \Pi(y) = \sum_x \Pi(x) P(x, y)

Will consider if $\Pi$ is unique & exists \exists \text{ i.e. doesn't depend on } \Pi_0

if periodic: could have no stat. dist. or several

if reducible: could have lots of stat. dist.

Some stat. dists:
\begin{align*}
\frac{1}{2} & \quad \frac{1}{2} \\
(\frac{1}{2}, \frac{1}{2}) & \quad (0, 1) \quad (1, 0)...
\end{align*}

Important Thm every ergodic M.C. has unique stationary distribution