Sublinear time algorithms for coloring graphs

Graph Coloring

def a proper c-coloring of G
assigns color c_v from "palette" $\{1..c\}$
to each $v \in V$ s.t. $\forall (u,v) \in E \quad c_u \neq c_v$

- in general, NP-complete
- important case solvable in linear time "Δ+1 - coloring"

max degree Δ
$c = \Delta + 1$

runtime: $O(m)$

Greedy algorithm:

for each $u \in V$
assign c_u different from
all c_v for $v \in N(u)$

always exists since have $\leq \Delta$ nbrs + palette size is $\Delta+1$

Can we do better?

Greedy list coloring:

Initial palette

For all v, let $\mathcal{L}(v) = \{1, .., \Delta+1\}$

For each $v \in V$ (arbitrary order)

if $\mathcal{L}(v) = \varphi$ output FAIL

else $c_v \leftarrow$ any color in $\mathcal{L}(v)$

remove c_v from all nbrs u of v

runtime:

$$\sum_{v} \Big(\underbrace{\text{time to find color in } \mathcal{L}(v) + \sum_{u \in N_v} \text{time to remove } c_v \text{ from } \mathcal{L}(u)}_{O(N_v)} \Big)$$

$= O(m)$

Sublinear Time Algorithm

↑ in m

not in n

Query model:

degree queries: what is $\deg(u)$?

pair queries: is $(u,v) \in E$?

nbr queries: what is k^{th} nbr of u?

Thm Can find $(\Delta+1)$-coloring in

$$\tilde{O}(n\sqrt{n}) \text{ time}$$

Comments

- no bound required on Δ for runtime
- non-adaptive
- $\Omega(n\sqrt{n})$ time required

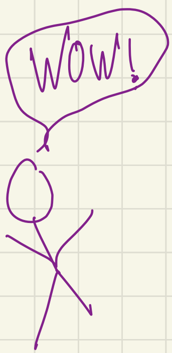
Palette Sparsification

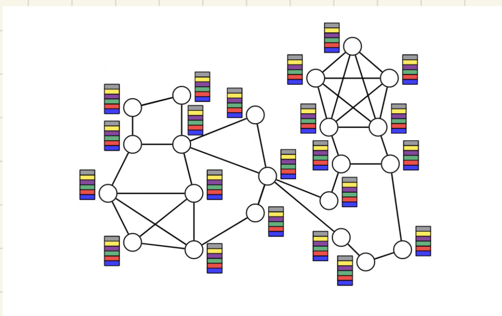
$\mathcal{L}(v)$ is v's sparser palette

$\forall$ nodes v,
sample $k = O(\log n)$ colors $\mathcal{L}(v)$ from $\{1, \dots, \Delta+1\}$

Surprisingly:
Sparsifying the palette this way (probably) doesn't kill colorability!

Main Claim whp, G can be colored s.t. $\forall v$, $c_v \in \mathcal{L}(v)$





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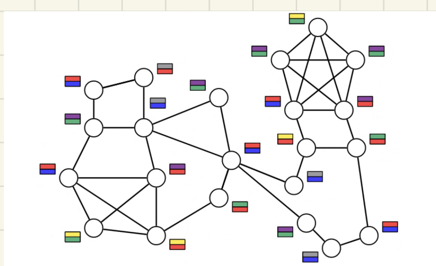
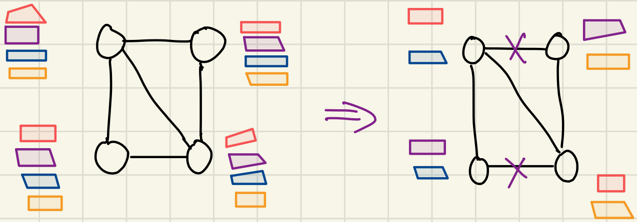


simple argument that always exists a color left to use for v in greedy no longer holds!!



can ignore "x"ed edges since won't cause each other to lose colors from their palette

Why palette sparsification?

idea (since guaranteed that coloring respecting smaller palettes remains)

$$\text{can } \underbrace{\text{throw out all } (u,v)}_{\text{sparsify edges in } G!!} \text{ s.t. } \underbrace{L(u) \cap L(v) \neq \emptyset}_{\text{they will not even consider using same colors!}}$$

how much sparser?

whp $O(n \log^2 n)$ edges remain

why? $\forall u \in V$, let $c_1 \cdots c_K$ be colors chosen by u

$\forall v \in N(u)$ & $i \in [1..K]$

$$\text{set } X_{v,i} \leftarrow \begin{cases} 1 & \text{if } v \text{ chooses color } c_i \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Let } X \leftarrow \sum_{i=1}^{K} \underbrace{\sum_{v \in N(u)} X_{v,i}}_{\text{\# edges due to color } c_i}$$

$\}$ upper bnd on $\deg(u)$ in sparsified graph since v might share 2 colors with u

$$E[X_{v,i}] = \Pr[X_{v,i}=1] = \frac{K}{\Delta+1}$$ ← v chose K out of $\Delta+1$ colors too what is probability it landed on i?

Then $$E[X] = \sum_{i=1}^{k} \sum_{v \in N(u)} E[X_{v,i}]$$ linearity of expectation

$$\leq K \cdot \Delta \cdot \frac{K}{\Delta+1} \leq K^2$$

Using $k = \Theta(\log n)$ shows $\forall u$,

expected degree of u in remaining graph is $O(\log^2 n)$.

Can show whp with more work ■

palette sparsification $\Rightarrow$ sublinear time!

(also, good sublinear space "streaming" algorithms

+ massively parallel computation algorithms)

$\tilde{O}(n^2/\Delta)$-time Palette Coloring Algorithm

time

1. Construct palette $\forall\, u \in V$ $\quad O(n \log n)$
2. Construct G_{sparse}: $\quad O(n \log n)$

$\forall c$, find $X_c = \{ v \mid c \in \mathcal{L}(v) \}$

query all pairs of nodes in each X_c to find E_{sparse}

how much time?

$$\underbrace{\text{query all pairs of nodes in each } X_c \text{ to find } E_{sparse}}_{\text{how much time?}}$$

$$\leq (\#\text{ colors } c) \times \underbrace{E\left[\binom{|X_c|}{2}\right]} \leq (\Delta+1)\, O\left(\frac{n^2 \log^2 n}{\Delta^2}\right)$$

$$E\left[\binom{|X_c|}{2}\right] \leq E\Big[\sum_{u,v \in V^2} 1_{u \,\&\, v \text{ pick color } c}\Big] = \binom{n}{2} \cdot \left(\frac{k}{\Delta+1}\right)^2 = O\left(\frac{n^2 \log^2 n}{\Delta^2}\right)$$

$$\leq \tilde{O}\left(\frac{n^2}{\Delta}\right)$$ (can be shown whp)

3. perform greedy list coloring problem on G_{sparse}

Greedy list coloring: time: $O(|E_{sparse}|) = \tilde{O}(n)$

For each $v \in V$ (arbitrary order)

if $L(v) = \phi$ output FAIL

else $c_v \leftarrow$ any color in $L(v)$

remove c_v from all nbrs u of v in G_{sparse}

Corr can find $\Delta+1$-coloring in $\tilde{O}(n^{3/2})$ time

why?

if $\Delta \le \sqrt{n}$

run greedy in $O(n\Delta) \le O(n\sqrt{n})$ time

if $\Delta > \sqrt{n}$

run palette sparsification in

$$\tilde{O}\left(\frac{n^2}{\Delta}\right) = \tilde{O}\left(\frac{n^2}{\sqrt{n}}\right) = \tilde{O}(n^{3/2})$$

time

$\blacksquare$

Why does main thm hold?

e.g. why is there still a $\Delta+1$ coloring after sparsification?

will show a weaker thm (allow 2Δ colors)

$\forall$ nodes v,
sample $O(\log n)$ colors $\mathcal{L}(v)$ from $\{1, \ldots, 2\Delta\}$

<u>Weaker Claim</u> whp, G can be colored s.t. $\forall v$, $c_v \in \mathcal{L}(v)$

<u>Pf</u> run Palette-coloring alg above with palette size S:
fail if a node ever runs out of colors

<u>Gain:</u>
much weaker than saying no previous nbr had c in its list

when attempt to color node v:
say color $c \in \{1..2\Delta\}$ "good" if c not used to color any nbr of v when v is considered

if $\mathcal{L}(v)$ contains any "good" color c, then can color v successfully

Since $\mathcal{L}(v)$ chosen independently of other lists, can think of choosing it "now". Since v has $\leq \Delta$ nbrs,

$\Pr[\mathcal{L}(v)$ contains no good color$]$

$\Pr[c \text{ bad}] \leq \dfrac{\Delta \text{ bad colors}}{2\Delta \text{ total colors}}$

$$\leq \left(\frac{\Delta}{2\Delta}\right)^s \underset{(*)}{\leq} \left(\frac{1}{2}\right)^s = \frac{1}{n^c}$$

Choosing $s = \Theta(\log n)$
$= c \log n$

+ union bnd for all $v \in V$

$\Longrightarrow$ whp algorithm never FAILS

$\Longrightarrow$ whp G has legal list coloring

why is $(*)$ true?

$$\Pr[\text{all } c \text{ bad}] \leq \frac{\binom{\Delta}{k}}{\binom{2\Delta}{k}} = \frac{\frac{(\Delta)(\Delta-1)\cdots(\Delta-k+1)}{k!}}{\frac{(2\Delta)(2\Delta-1)\cdots(2\Delta-k+1)}{k!}} < \frac{1}{2}^k$$