Monotonicity testing

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6.5240 Sublinear Time Algorithms

(slides on testing monotonicity of functions $f: \{0,1\}^n \rightarrow \{0,1\}$ from Sofya Raskhodnikova)

Alphabetized?

https://petapixel.com/2017/11/14/photos-new-futuristic-library-china-1-2-million-books/

Sortedness of a sequence

- Given: list $y_1\, y_2 \ldots y_n$
- Question: is the list sorted?

- Clearly requires *n* steps – must look at each y_i

Sortedness of a sequence

- Given: list $y_1\, y_2 \ldots\, y_n$
- Question: can we quickly test if the list close to sorted?

What do we mean by ``quick''?

- query complexity measured in terms of list size *n*

- Our goal (if possible):
  - *Very small* compared to *n,* will go for *clog n*

What do we mean by "close''?

Definition: a list of size n is ε-close to sorted if can delete at most εn values to make it sorted. Otherwise, ε-far.

(ε is given as input, e.g., ε=1/5)

Sorted:	1	2	4	5	7	11	14	19	20	21	23	38	39	45
Close:	1	4	2	5	7	11	14	19	20	39	23	21	38	45
	1	4		5	7	11	14	19	20		23		38	45
Far:	45	39	23	1	38	4	5	21	20	19	2	7	11	14
				1		4	5					7	11	14

Requirements for algorithm:

- Pass sorted lists
- Fail lists that are ε-far.
  - Equivalently: if list likely to pass test, can change at most ε fraction of list to make it sorted

What if list not sorted, but not far?

Probability of success > ¾

(can boost it arbitrarily high by repeating several times and outputting "fail" if ever see a "fail", "pass" otherwise)

- Can test in $O(1/\varepsilon \log n)$ time

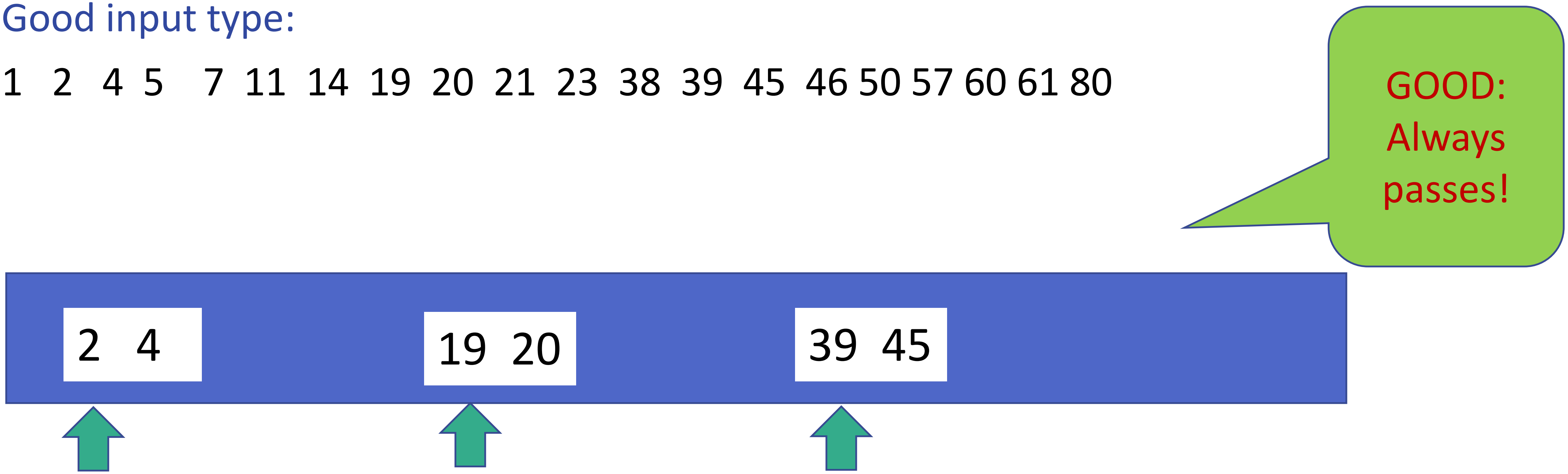
(and can't do any better!)

A first try for an algorithm:

Pick *random entry* and test that *entry and its right neighbor* are in the *correct order*

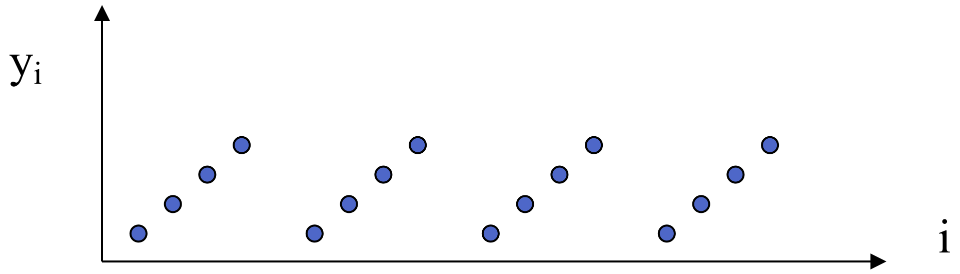
Good input type:

1 2 4 5 7 11 14 19 20 21 23 38 39 45 46 50 57 60 61 80



First try (cont.):

- Proposed algorithm:
  - Pick random *i* and test that $y_i \le y_{i+1}$
- Bad input type:
  - 1,2,3,4,5,…n/4, 1,2,….n/4, 1,2,…n/4, 1,2,…,n/4
  - Difficult for this algorithm to find “breakpoint”
  - But other tests work well on this input…

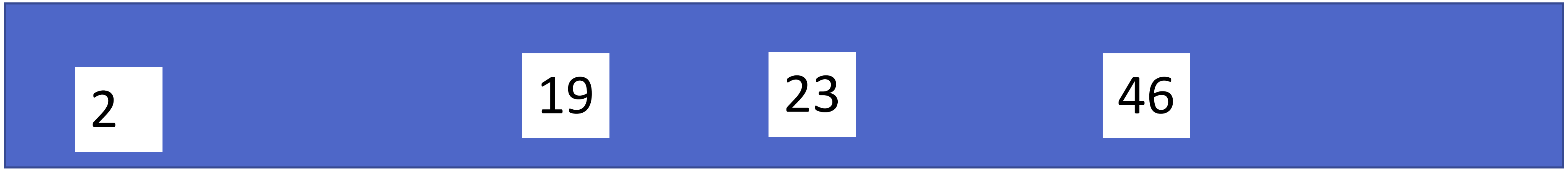


A second try for an algorithm:

Pick lots of random entries and pass if all in right order

Good input type:

1 2 4 5 7 11 14 19 20 21 23 38 39 45 46 50 57 60 61 80

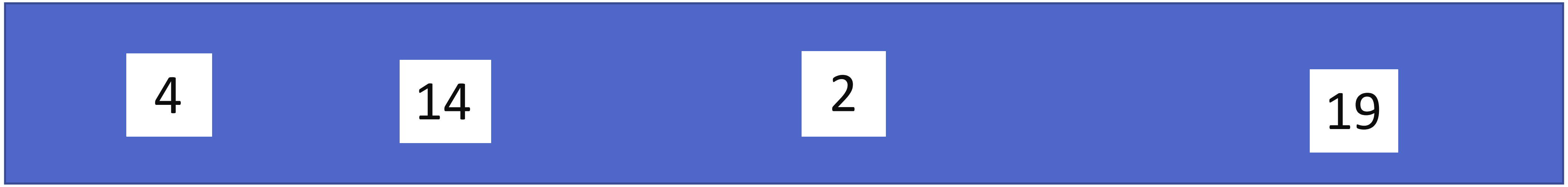


A second try:

Pick lots of random entries and pass if all in right order

Bad input type:

1 2 4 5 7 11 14 19 20 21 1 2 4 5 7 11 14 19 20 21



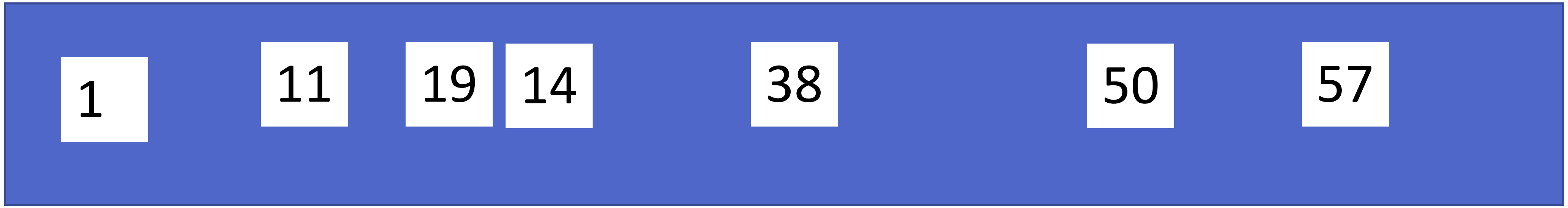
A second try:

Pick lots of random entries and pass if all in right order

How many?

Another bad input type:

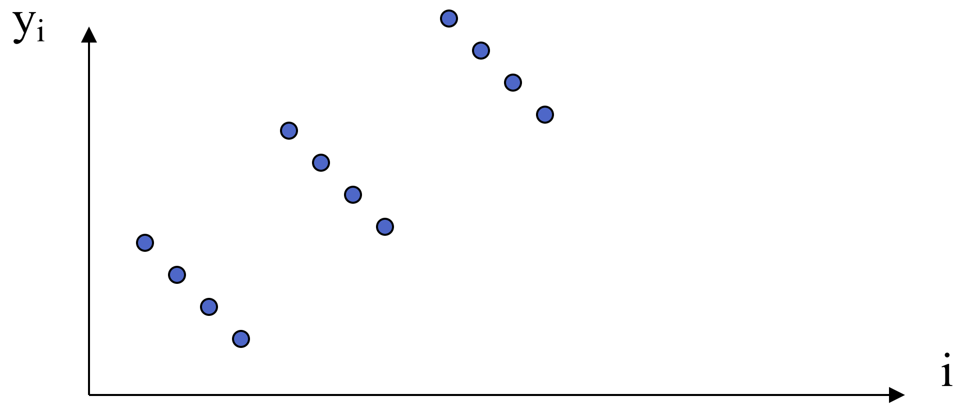
2 1 5 4 11 7 19 14 21 20 38 23 45 39 50 46 60 57 80 61



A second attempt:

- Proposed algorithm:
  - Pick random $i<j$ and test that $y_i \leq y_j$
- Bad input type:
  - $n/4$ groups of 4 decreasing elements

    4,3, 2, 1,8,7,6,5,12,11,10,9…,4k, 4k-1,4k-2,4k-3,…
  - Largest monotone sequence is n/4
  - must pick i,j in same group to see problem
  - need $\Omega(n^{1/2})$ samples. (also $O(n^{1/2})$ is enough)



A minor simplification:

- Assume list is distinct (i.e. $x_i \neq x_j$)

- Claim: this is not really easier
  - Why?

    Can "virtually" append *i* to each x_i

    $x_1,x_2,\ldots x_n \rightarrow (x_1,1), (x_2,2),\ldots,(x_n,n)$

    e.g., $1,1,2,6,6 \rightarrow (1,1),(1,2),(2,3),(6,4),(6,5)$

    Breaks ties without changing order

A test that works

- The test:

  Test $O(1/\varepsilon)$ times:
    - Pick random i
    - Look at value of y_i
    - Do binary search for y_i
    - Does the binary search find y_i at location i? If not, FAIL
    - Does the binary search find any inconsistencies? If yes, FAIL
    - Do we end up at location i? If not FAIL

  Pass if never failed

- Running time: $O(\varepsilon^{-1} \log n)$ time
- Why does this work?

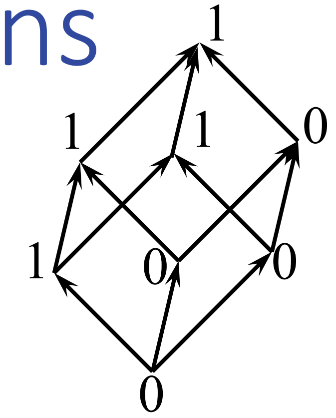
Behavior of the test:

- Define index *i* to be good if binary search for y_i successful
- O(1/ε log n) time test (restated):
  - pick *O*(1/ε) *i*'s and pass if they are all good
- Correctness:
  - If list is sorted, then all i's good (uses distinctness) → test always passes
  - If list likely to pass test, then at least (1-ε)*n* i's are good.
    - Main observation: good elements form increasing sequence
      - Proof: for i<j both good need to show $y_i < y_j$
        - let k = least common ancestor of i,j
        - Search for i went left of k and search for j went right of k → $y_i < y_k < y_j$
    - Thus list is ε-close to monotone (delete < ε*n* bad elements)

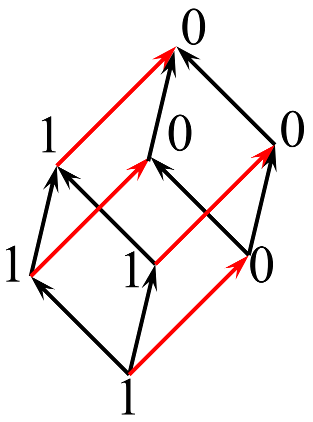
Monotonicity of Functions

[A function $f : \{0,1\}^n \to \{0,1\}$ is monotone
if increasing a bit of x does not decrease $f(x)$.

- Is f monotone or ε-far from monotone
  (f has to change on many points to become monontone)?
  - Edge $x \to y$ is violated by f if $f(x) > f(y)$.



monotone



$\frac{1}{2}$-far from monotone

Time:

- Today: $O(n/\varepsilon)$, logarithmic in the size of the input, 2^n
- Newer: $\Theta(\sqrt{n}/\varepsilon^2)$ for nonadaptive tests, $\Omega\left(n^{\frac{1}{3}}\right)$

Monotonicity Test

Idea: Show that functions that are far from monotone violate many edges.

EdgeTest (f, ε)

1. Pick $2n/\varepsilon$ edges (x, y) uniformly at random from the hypercube.
2. **Reject** if any (x, y) is violated (i.e. $f(x) > f(y)$). Otherwise, **accept**.

Analysis

- If f is monotone, EdgeTest always accepts.
- If f is ε-far from monotone, will show that $\geq \varepsilon/n$ fraction of edges (i.e., $\frac{\varepsilon}{n} \cdot 2^{n-1}n = \varepsilon 2^{n-1}$ edges) violated by f.
  - Let $V(f)$ denote the number of edges violated by f.

Contrapositive: If $V(f) < \varepsilon\, 2^{n-1}$,
f can be made monotone by changing $< \varepsilon\, 2^n$ values.

Repair Lemma

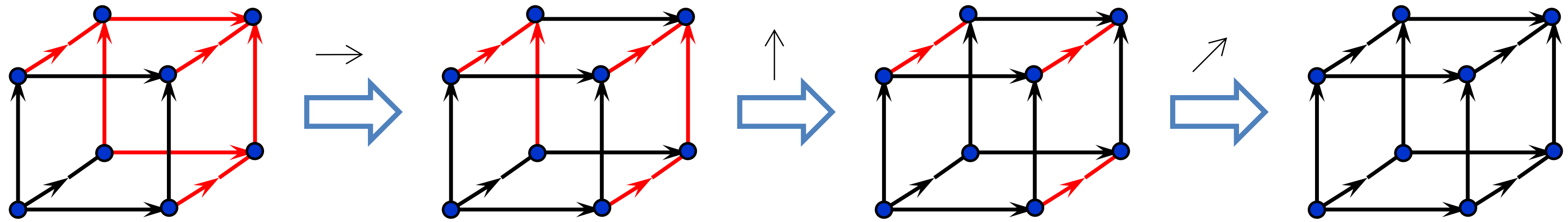
f can be made monotone by changing $\leq 2 \cdot V(f)$ values.

Repair Lemma: Proof Idea

Repair Lemma

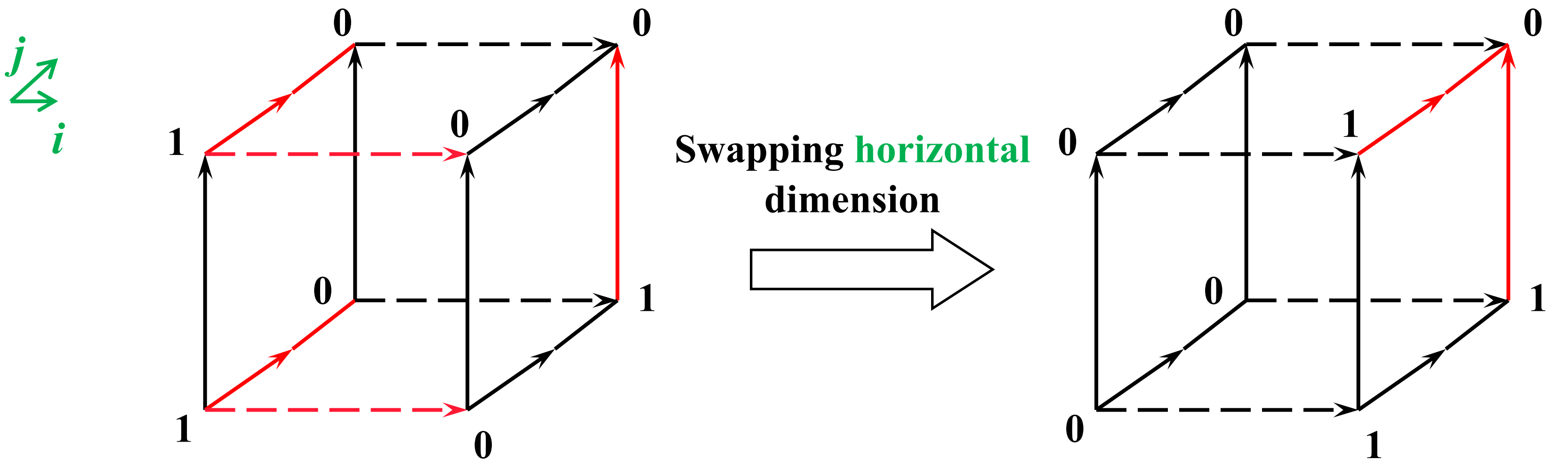
f can be made monotone by changing $\leq 2 \cdot V(f)$ values.

Proof idea: Transform f into a monotone function by repairing edges in one dimension at a time.



Repairing Violated Edges in One Dimension

Swap violated edges *1→0* in one dimension to *0→1*.



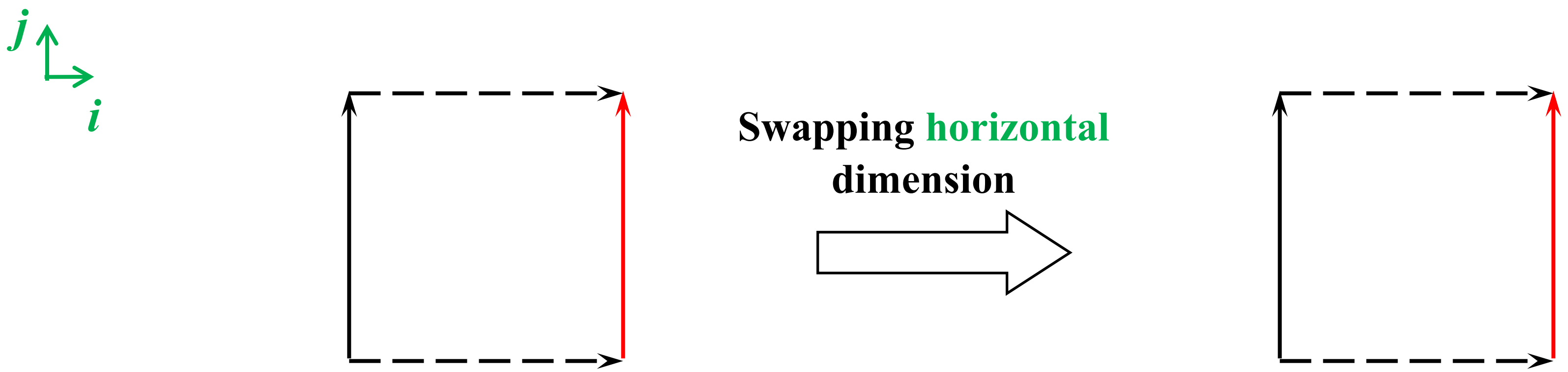
Let V_j = # of violated edges in dimension j

Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$

Enough to prove the claim for squares

Proof of The Claim for Squares

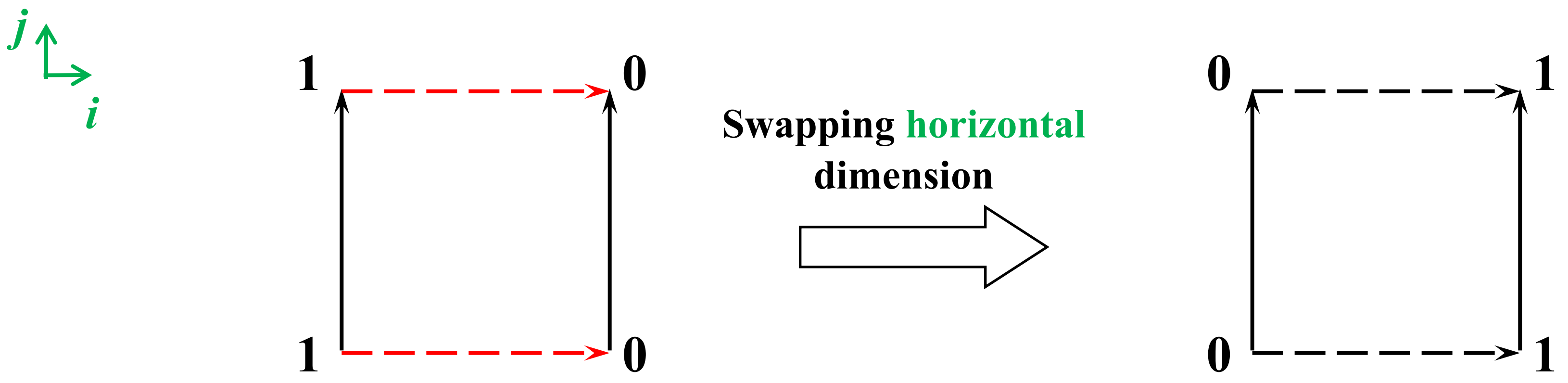
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- If no horizontal edges are violated, no action is taken.

Proof of The Claim for Squares

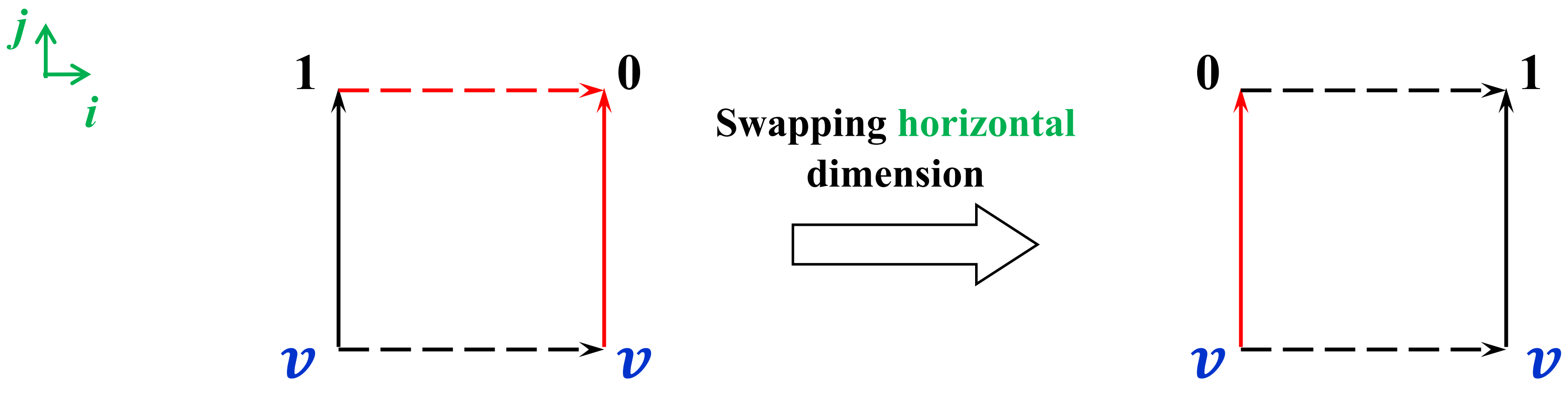
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- If both horizontal edges are violated, both are swapped, so the number of vertical violated edges does not change.

Proof of The Claim for Squares

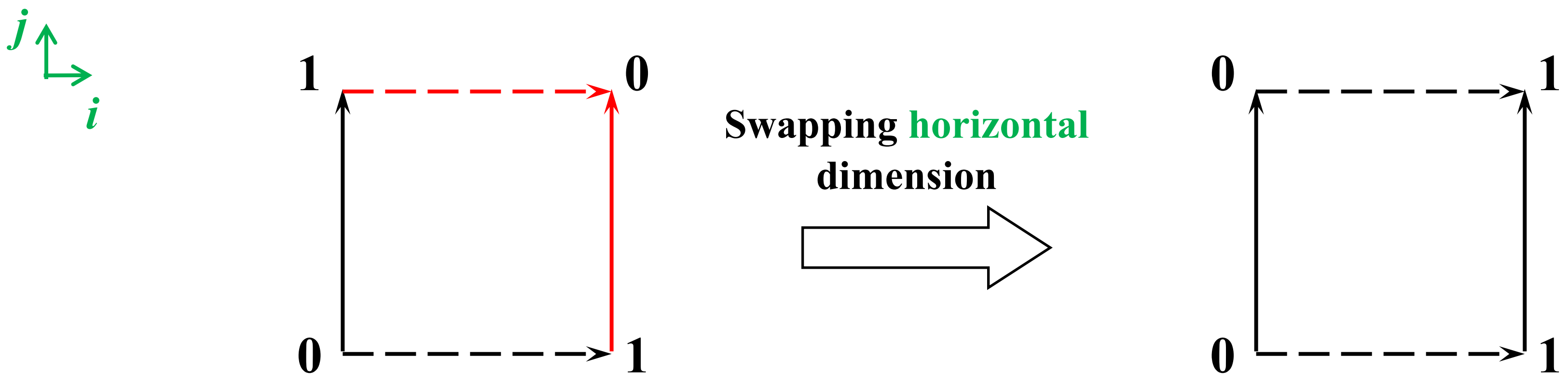
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- Suppose one (say, top) horizontal edge is violated.
- If both bottom vertices have the same label, the vertical edges get swapped.

Proof of The Claim for Squares

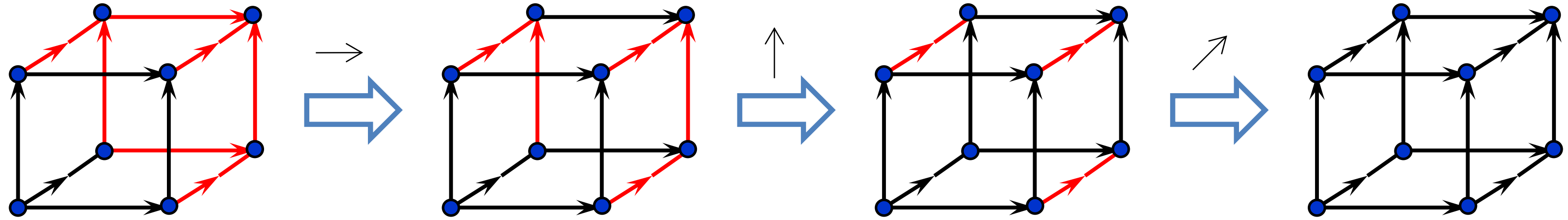
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- Suppose one (say, top) horizontal edge is violated.
- If both bottom vertices have the same label, the vertical edges get swapped.
- Otherwise, the bottom vertices are labeled 0→1, and the vertical violation is repaired.

Proof of The Claim for Squares

Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



After we perform swaps in all dimensions:

- f becomes monotone
- # of values changed:

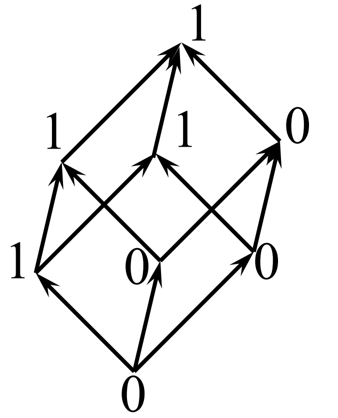
$$2 \cdot V_1 + 2 \cdot (\text{\# violated edges in dim 2 after swapping dim 1}) + 2 \cdot (\text{\# violated edges in dim 3 after swapping dim 1 and 2}) + \ldots \leq 2 \cdot V_1 + 2 \cdot V_2 + \cdots 2 \cdot V_n = 2 \cdot V(f)$$

Repair Lemma

f can be made monotone by changing $\leq 2 \cdot V(f)$ values.

- Can improve the bound by a factor of 2.

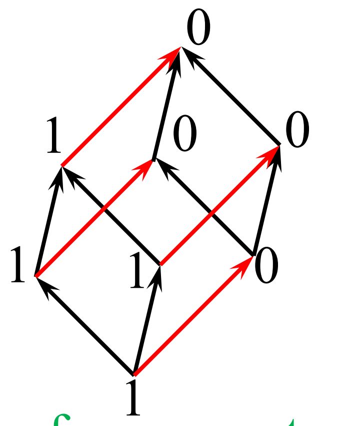
Testing if a Functions $f : \{0,1\}^n \rightarrow \{0,1\}$ is monotone



monotone

Monotone or

ε-far from monotone?



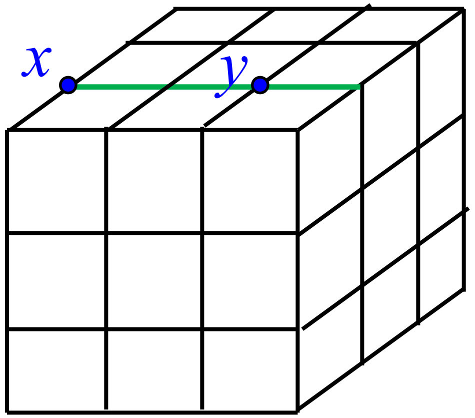
$\frac{1}{2}$-far from monotone

O(n/ε) time

(logarithmic in the size of the input)

Testing Properties of High-Dimensional Functions

In polylogarithmic time, we can test a large class of properties of functions $f: \{1, \dots, n\}^d \to \mathbb{R}$, including:



- Lipschitz property
- Bounded-derivative properties
- Unateness