6.895 Randomness and Computation	April 3, 2006
Lecture 14	
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1 Analysis of the Markov Chain

1.1 Using Congestion and Canonical Paths

How do we see that the transitions of the Markov Chain from lecture 13 has low congestion? We use a canonical path argument. Consider one edge, from matching M_a to M_b. Consider some canonical path, as specified in lecture 13, from matching M_1 to M_2 that uses this edge. Given the edge (And the direction we cross it), we only need a small amount of additional information to uniquely identify M_1 and M_2. The amount of information we need limits the number of canonical paths that cross the edge and thus provides a bound on congestion.

1.2 Identifying a Path

We will need some notation:
Let $M_1 \oplus M_2$ be the set of edges in exactly one of M_1, M_2.
Let $\overline{M} = (M_1 \oplus M_2) \setminus M_a$, though this is not necessarily a matching.

Claim 1 *We can reconstruct* (M_1, M_2) *from* $\left(\overline{M}, M_a, M_b\right)$.

- *Uncorrected edges in* M_a *match edges in* M_1*. (We know what has been corrected, since we know the lexicographical ordering.)*
- *Corrected edges in* $M_a \oplus \overline{M}$ *reveal the other edges of* M_1*.*
- *We can determine* M_2 *similarly from* M_B*.*

1.3 $\overline{M}$ is Almost a Matching

We will sidestep the issue of the number of bits needed to specify $\overline{M}$ by bounding the congestion in terms of n; the size of the Markov Chain. We notice that although $\overline{M}$ is not a matching, it can be transformed into a matching by removing at most 2 edges, due to the structure of the fixing procedure definted earlier.

With the 2 edges e_1 and e_2 such that $\overline{M} \setminus \{e_1, e_2\}$ is a matching, we can specify $\left(\overline{M}, M_a, M_b\right)$ with $\left(\overline{M} \setminus \{e_1, e_2\}, e_1, e_2, M_a, M_b\right)$.

1.4 Putting All the Pieces Together

Now, the matching $\overline{M}\setminus\{e_1, e_2\}$ can correspond to any state of the Markov Chain, while the edges could be any edges. Thus, any transition from M_a to M_b has congestion at most (# states in Markov Chain)$\cdot m^2$. We saw, from our previous analysis of the Canonical Path Technique, that if we have congestion of (# states in Markov Chain) $\cdot\, \alpha$ in a d-regular graph, then the conductance, Φ_G is $\geq \frac{1}{d \cdot \alpha}$. Thus, the conductance here is at least $\frac{1}{m^3}$.

1.5 Determining the Mixing Time

Again from the previous lecture, in order to mix we will need a number of steps equal to

$$t = \frac{4}{\left(\frac{1}{m^3}\right)^2} \ln\left(\frac{2}{\epsilon^2} \cdot (\#\text{ matchings})\right)$$
$$= 4m^6 \ln\left(\frac{2}{\epsilon^2} \cdot (\#\text{ matchings})\right)$$

$\leq 4m^6 \ln\left(\frac{2}{\epsilon^2} \cdot 2^m\right)$
$\leq O\left(m^7 \ln\left(\frac{1}{\epsilon^2}\right)\right)$

1.6 Conclusions for the Markov Chain

Thus we need only a number of steps polynomial in the size of the graph and the reciprocal of the error (although this is a large polynomial). The Markov Chain we defined in the last class will mix rapidly enough to provide near-uniform generation.

2 Relating Linear Algebra to Mixing

2.1 Graphs and Linear Algebra

Given an undirected, d-regular graph G, let P be the transition matrix of G. P is both real and symmetric. Recalling our Linear Algebra, we say that v is an *eigenvector* of P with *eigenvalue* λ if $vP = \lambda v$.

Theorem 2 *If P is real and symmetric, then $\exists$ an orthonormal basis $v^{(1)}, \ldots, v^{(n)}$ of eigenvectors of P with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. We will label these so that $|\lambda_i| \geq |\lambda_{i+1}|$.*

Example: The transition matrix, P, of a random walk on a d-regular graph has eigenvector $\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$ with eigenvalue 1.

Next time: We will review more Linear Algebra and look at relating eigenvalues of P with the mixing time of G.