1 Recap from Last Time

Definition 1 (Fourier cofficient) $\hat{f}(s) = \langle f, \chi_s \rangle = \frac{1}{2^n} \sum\limits_{x \in \{\pm 1\}^n} f(x)\chi_s(x)$

Theorem 1 $\forall_{f,x}\ f(x) = \sum\limits_{s \subset [n]} \hat{f}(s)\chi_s(x)$

Last time, we learned that the magnitute of the Fourier coefficient is in agreement with the linear function. Based on this, we can use Fourier analysis to test linearity.

2 Uses of Fourier Analysis

Now, we are going to use Fourier analysis to:

1. estimate sums of powers of degree 1 Fourier coefficient of f
2. estimate the maximum of degree 1 Fourier coefficient

2.1 Estimate sums of powers

We would like to estimate sums of powers of degree 1 Fourier coefficients of f, given oracle access to f. (We are only inerested in sets of size 1, and thus "degree 1".)

$$\sum_{|s|=1} \hat{f}(s)^p \qquad \text{(for } p \geq 2\text{)}$$

The algorithm is a random process:

- Pick $x^{(1)}, x^{(2)}, \ldots, x^{(p-1)} \in_{\mathbb{R}} \{\pm 1\}^n$ uniformly
- Pick noise vector μ such that entry $\begin{matrix} +1 \\ -1 \end{matrix}$ with probability $\begin{matrix} \frac{1}{2} + \frac{\eta}{2} \\ \frac{1}{2} - \frac{\eta}{2} \end{matrix}$
- $y \leftarrow f(x^{(1)})f(x^{(2)}) \cdots f(x^{(p-1)})f(x^{(1)} \odot x^{(2)} \cdots x^{(p-1)} \odot \mu)$ (note that for $p = 3$, it almost looks like a linearity test with noise)

We will show: $\mathrm{E}[y] = \sum\limits_{s \subseteq [n]} \eta^{|s|}\hat{f}(s)^p$, which will look a lot like the analysis of the linearity test last time.

Proof

$$\mathrm{E}[y] = \mathrm{E}\left[\sum_{s_1, s_2, \ldots, s_p} \hat{f}(s_1)\hat{f}(s_2) \cdots \hat{f}(s_{p-1})\hat{f}(s_p)\chi_{s_1}(x^{(1)}) \cdots \chi_{s_{p-1}}(x^{(p-1)})\chi_{s_p}(x^{(1)} \odot x^{(2)} \cdots x^{(p-1)} \odot \mu)\right]$$

Since

$$\chi_{s_p}(x^{(1)} \odot x^{(2)} \cdots x^{(p-1)} \odot \mu) = \chi_{s_p}(x^{(1)} \odot x^{(2)} \cdots x^{(p-1)}) \cdot \chi_{s_p}(\mu) = (\chi_{s_p}(x^{(1)}) \cdot \chi_{s_p}(x^{(2)}) \cdots \chi_{s_p}(x^{(p-1)})) \cdot \chi_{s_p}(\mu),$$

then

$$\mathrm{E}\,[y] = \sum_{s_1,s_2,\ldots,s_p} \hat{f}(s_1)\cdots\hat{f}(s_p)\mathrm{E}\left[\chi_{s_1\triangle s_p}(x^{(1)})\chi_{s_2\triangle s_p}(x^{(2)})\cdots\chi_{s_{p-1}\triangle s_p}(x^{(p-1)})\chi_{s_p}(\mu)\right]$$

$$= \sum_{s_1,s_2,\ldots,s_p} \hat{f}(s_1)\cdots\hat{f}(s_p)\mathrm{E}\left[\chi_{s_1\triangle s_p}(x^{(1)})\right]\mathrm{E}\left[\chi_{s_2\triangle s_p}(x^{(2)})\right]\cdots\mathrm{E}\left[\chi_{s_{p-1}\triangle s_p}(x^{(p-1)})\right]\mathrm{E}\left[\chi_{s_p}(\mu)\right]$$

The last line follows since the terms in the expectation are independent.

- if $s_1 = s_2 = \cdots = s_p$, $s_i \triangle s_p = \emptyset$, then

  $$\mathrm{E}\left[\chi_{s_i\triangle s_p}(x^{(i)})\right] = 1$$

  Also $\mathrm{E}\,[\chi_{s_p}(\mu)] = \eta^{|s_p|}$ since:

  $$\mathrm{E}\,[\mu_i] = 1\cdot(1+\frac{\eta}{2}) + (-1)\cdot(1-\frac{\eta}{2}) = \eta$$

  $$\mathrm{E}\,[\prod_{i\in s_p}\mu_i] = \prod_{i\in s_p}\mathrm{E}\,[\mu_i] = \eta^{|s_p|} \implies \mathrm{E}\,[\chi_{s_p}(\mu)] = \eta^{|s_p|}$$

- otherwise, some $s_i \triangle s_p \neq \emptyset$. In this case, using reasoning as in the last lecture, $\mathrm{E}\,[\chi_{s_i\triangle s_p}] = 0$ so, the entire product equals 0.

Therefore, $\mathrm{E}\,[y] = \sum_s \hat{f}(s)^p \eta^{|s|}$

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We want to estimate $\sum_{|s|=1} \hat{f}(s)^p$ to an additive $\pm\eta^2$, so we do the following:

1. Estimate $\mathcal{A} \equiv \mathrm{E}\,[f(x^{(1)}f(x^{(2)}))\cdots f(x^{(p)})] = \hat{f}(\emptyset)^p$
2. Estimate $\mathcal{B} \equiv \mathrm{E}\,[f(x^{(1)})f(x^{(2)}))\cdots f(x^{(p)})f(x^{(1)}\odot x^{(2)}\odot\cdots\odot x^{(p-1)}\odot\mu)] = \sum_s \eta^s\hat{f}(s)^p$

Using the additive version of Chernoff bound: for var $x\in[-1,1]$, to additive $\pm\varepsilon$ needs $O(\frac{1}{\varepsilon^2})$ queries $\Rightarrow$ we need $O(\frac{1}{\eta^4})$ queries. However, we only got rid of one term, *i.e.* $\mathcal{B}-\mathcal{A} = \sum_{|s|>0}\eta^{|s|}\hat{f}(s)^p \equiv \mathcal{C}$, and we want to estimate to witin $\pm\eta^2$, so we will show that $\frac{\mathcal{C}}{\eta}$ is already a good approximate of $\sum_{|s|=1}\hat{f}(s)^p$ (with additive error η).

Claim 2 $\frac{\mathcal{C}}{\eta}$ *is a good estimate of* $\sum_{|s|=1}\hat{f}(s)^p$

Proof of Claim

$$\sum_{|s|=1}\eta\hat{f}(s)^p = \sum_{|s|>0}\eta^{|s|}\hat{f}(s)^p - \sum_{|s|>1}\eta^{|s|}\hat{f}(s)^p$$

$\sum_{|s|>0}\eta^{|s|}\hat{f}(s)^p = \mathcal{C}$, so we want to claim that $\sum_{|s|>1}\eta^{|s|}\hat{f}(s)^p$ is not too big, *i.e.* $\eta^{|s|}$ is exponentially decayed:

$\sum_{|s|>1}\eta^{|s|}\hat{f}(s)^p \le \eta^2\sum_{|s|>1}|\hat{f}(s)^p|$, or we can say $\sum_{|s|>1}\eta^{|s|}\hat{f}(s)^p \le \eta^2\sum_{|s|>1}\hat{f}(s)^2$ since f is boolean, $|\hat{f}(x)\le 1|$.

Thus, by Parseval's theorem in the Boolean case, $\sum_{|s|>1} \hat{f}(s)^2 \leq 1 \Rightarrow \sum_{|s|>1} \eta^{|s|}\hat{f}(s)^p \leq \eta^2$.

So, $\sum_{|s|=1} \hat{f}(s)^p = \frac{\mathcal{C}}{\eta} \pm \frac{\eta^2}{\eta} \Rightarrow \eta$-additive $\Rightarrow \frac{\mathcal{C}}{\eta}$ to within $\pm\eta$.

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Theorem 3 *For $p \geq 2$, we can estimate $\sum_{|s|=1} \hat{f}(s)^p$ to within additive $\pm\eta$ with $O(\frac{p}{\eta^4})$ queries.*

2.2 Estimate the maximum

We can also use Fourier analysis to estimate the maximum of degree 1 Fourier coefficient. (Note that this is a weak estimate.)

If $\forall_i \hat{f}(\{i\}) < \alpha$, then $\sum_{|s|=1} \hat{f}(s)^4 < \sum_{|s|=1} \alpha^2 \hat{f}(s)^2 < \alpha^2 \cdot 1 \qquad (*)$

(Parseval when f is boolean)

Definition 2 *f is τ-regular if $|\hat{f}(\{i\})| < \tau \quad \forall_i$*

Theorem 4 *$\exists O(\frac{1}{\tau^{16}})$-query test with following behavior:*

- *if $\exists i$ such that $|\hat{f}(\{i\})| \geq \tau$ then $\Pr[reject] \geq \frac{3}{4}$ (not τ-regular)*
- *if $\forall_i |\hat{f}(\{i\})| < \frac{\tau^2}{4}$ then $\Pr[accept] \geq \frac{3}{4}$ ($\frac{\tau^2}{4}$-regular)*

Proof We estimate $\sum_i \hat{f}(\{i\})^4$ to within $\frac{\tau^4}{4}$, and accept iff estimate $\geq \frac{\tau^4}{2}$.

If $\exists i$ such that $|\hat{f}(\{i\})| \geq \tau$, then $\sum \hat{f}(\{i\})^4 \geq \tau^4$. Estimate $\geq \frac{3}{4}\tau^4$, so will reject.
If $\forall_i |\hat{f}(\{i\})| \leq \frac{\tau^2}{4}$, then by $(*)$, $\sum \hat{f}(\{i\})^4 \leq \frac{\tau^4}{16}$. Estimate $\leq \frac{5}{16}\tau^4 < \frac{1}{2}$ and will accept.

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3 LTF: Linear Threshold Function

$f(x) = \operatorname{sgn}(w_1 x_1 + \cdots + w_n x_n + \theta)$, where

$x_i \in \{\pm 1\}, w_i \in \mathbb{R}, \theta \in \mathbb{R}, \operatorname{sgn}(x) = \begin{cases} +1 & \text{if} \quad x \geq 0 \\ -1 & \phantom{\text{if}} \quad x < 0 \end{cases}$

There is a constant time tester independent of n, but we will only look at a special case here: τ-regular balanced LTF's ("balanced" means that $\theta = 0$)

Theorem 5 *$\exists$ testing algorithm $\mathbf{A}$ that requires $O(\frac{1}{\tau^8})$ queries:*

- *If f is a balanced τ-regular LTF, then $\Pr[\mathbf{A}$ passes$] \geq \frac{3}{4}$*
- *If $C\tau^{1/6}$-far from any balanced τ-regular LTF, $\Pr[\mathbf{A}$ fails$] \geq \frac{3}{4}$ (C is a constant that will be determined later)*

The high-level algorithm works this way:

1. Estimate $\sum_{|s|=1} \hat{f}(s)^4$ to additive $\pm\tau^2$, if $> 2\tau^2$, fail
2. Estimate $\sum_{|s|=1} \hat{f}(s)^2$ to additive $\pm C_1\tau^{1/3}$ (C_1 is another constant), if within $\pm 2C_1\tau^{1/3}$ of $\frac{2}{\Pi}$, output pass; else fail

(To be continued in the next lecture ...)