1. Given a boolean function $f(\cdot)$ on boolean inputs, a sequence $C = C_1, C_2, \ldots$ of circuits is a circuit family for $f(\cdot)$ if $C_n$ has $n$ inputs and computes $f(x_1, \ldots, x_n)$ at its output for all $n$ bit inputs $(x_1, \ldots, x_n)$. The family $C$ is said to be polynomial-sized if the size of $C_n$ is bounded above by $p(n)$ for every $n$, where $p(\cdot)$ is a polynomial. A randomized circuit family for $f(\cdot)$ is a circuit family for $f(\cdot)$ that, in addition to the $n$ inputs $x_1, \ldots, x_n$, takes $m$ inputs $r_1, \ldots, r_m$, each of which is equiprobably and independently 0 or 1. In addition, for every $n$, circuit $C_n$ must satisfy

(a) if $f(x_1, \ldots, x_n) = 0$ then output 0 regardless of the values of the random inputs $r_1, \ldots, r_m$.
(b) if $f(x_1, \ldots, x_n) = 1$ then output 1 with probability $\geq 1/2$.

Show: If a boolean function has a randomized polynomial sized circuit family, then it has a polynomial sized circuit family.

2. You are given a 2-SAT formula $\phi(x_1, \ldots, x_n)$. Consider the following algorithm for finding a satisfying assignment:

- Start with an arbitrary assignment. If it’s satisfying, output it and halt.
- Do $s$ times:
  - Pick an arbitrary unsatisfied clause
  - Pick one of the two literals in it uniformly at random
  - Complement the setting of the chosen literal
  - If the new assignment satisfies $\phi$, output the assignment and halt.

Show that if you pick $s$ to be $O(n^2)$, you will output a satisfying assignment with probability at least $3/4$.

3. Let $G(V, E)$ be a graph with $n$ vertices such that for some constant $\alpha > 0$, and every set $S \subseteq V$ with $n/2$ vertices,

$$|\{w \in V | \exists v \in S, (v, w) \in E\}| \geq \frac{n}{2} + \alpha n.$$ 

For any positive integer $k$, let $W_1, \ldots, W_k$ be subsets of $V$ of size at least $(1 - \alpha)n$ each. Show that there exists a path $(v_1, \ldots, v_k)$ in $G$ such that for $1 \leq i \leq k$, $v_i \in W_i$.

4. A $d$-regular graph has $(K, A)$-vertex expansion if $\forall S \subseteq V, |S| \leq K, |\lambda(S)| \geq A|S|$ where $\lambda(S) = \{|u| \exists v \in S s.t. (u, v) \in E\}$. For a probability distribution $\pi$ over $[n]$, the collision probability is

$$||\pi||^2 = \sum_x \pi_x^2$$

Show that the following is true:
• $||\pi||^2 \geq \frac{1}{|S(\pi)|}$ where $S(\pi) = \{x | \pi_x > 0\}$.
• $||\pi||^2 = ||\pi - u||^2 + \frac{1}{n}$ where $u$ is the uniform distribution.
• If there is a constant $\lambda < 1$ such that the transition matrix of $G$ is such that $|\lambda_2| \leq \lambda$ then for any $\alpha < 1$, $G$ has vertex expansion $(\alpha n, (1-\alpha)\lambda^2 + \alpha)$.

**Remark:** We use the convention that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the random walk matrix.

5. (optional) Let $G = (V, E)$ be a $d$-regular, $\alpha$-expander, i.e., $\forall S \subset V, |S| \leq |V|/2, |\lambda(S)\backslash S| \geq \alpha|S|$ ($\lambda$ is defined as above). Let $n = |V|$. Suppose a set of pairs $S = \{(a_1, b_1), \ldots, (a_q, b_q)\}$ are such that $a_i$ and $b_i$ are chosen uniformly and independently from $V$ for all $i$. Show that for all $a, b$, there is a path connecting $a, b$ of length $O(\log n)$. Show that there exists a way of connecting each $a_i, b_i$ pair via a path of length $O(\log n)$, such that no edge is used more than $O(\log n)$ times in total over all paths.

**Hint:** for each $(a_i, b_i)$ pair, choose a random $x_i$. Show how to pick a “random” path from $a_i$ to $x_i$ and from $b_i$ to $x_i$. Show that your method is such that no edge is used more than $O(\log n)$ times in total.