6.842 Randomness and Computation	April 9, 2006

Homework 4

Lecturer: Ronitt Rubinfeld — *Due Date: April 23, 2006*

1. Given a boolean function $f(\cdot)$ on boolean inputs, a sequence $C = C_1, C_2, ...$ of circuits is a *circuit family for* $f(\cdot)$ if C_n has n inputs and computes $f(x_1, \ldots, x_n)$ at its output for all n bit inputs $(x_1, \ldots, x_n)$. The family C is said to be *polynomial-sized* if the size of C_n is bounded above by $p(n)$ for every n, where $p(\cdot)$ is a polynomial. A *randomized circuit family for* $f(\cdot)$ is a circuit family for $f(\cdot)$ that, in addition to the n inputs $x_1, \ldots, x_n$, takes m inputs $r_1, \ldots, r_m$, each of which is equiprobably and independently 0 or 1. In addition, for every n, circuit C_n must satisfy

   (a) if $f(x_1, \ldots, x_n) = 0$ then output 0 regardless of the values of the random inputs $r_1, \ldots, r_m$.

   (b) if $f(x_1, \ldots, x_n) = 1$ then output 1 with probability $\geq 1/2$.

   Show: If a boolean function has a randomized polynomial sized circuit family, then it has a polynomial sized circuit family.

2. You are given a 2-SAT formula $\phi(x_1, \ldots, x_n)$. Consider the following algorithm for finding a satisfying assignment:

   - Start with an arbitrary assignment. If it's satisfying, output it and halt.
   - Do s times:
     - Pick an arbitrary unsatisfied clause
     - Pick one of the two literals in it uniformly at random
     - Complement the setting of the chosen literal
     - If the new assignment satisfies ϕ, output the assignment and halt.

   Show that if you pick s to be $O(n^2)$, you will output a satisfying assignment with probability at least 3/4.

3. Let $G(V, E)$ be a graph with n vertices such that for some constant $\alpha > 0$, and every set $S \subseteq V$ with $n/2$ vertices,

$$|\{w \in V | \exists v \in S, (v, w) \in E\}| \geq \frac{n}{2} + \alpha n.$$

   For any positive integer k, let $W_1, \ldots, W_k$ be subsets of V of size at least $(1 - \alpha)n$ each. Show that there exists a path $(v_1, \ldots, v_k)$ in G such that for $1 \leq i \leq k$, $v_i \in W_i$.

4. A d-regular graph has (K, A)-vertex expansion if $\forall S \subset V, |S| \leq K, |\lambda(S)| \geq A|S|$ where $\lambda(S) = |\{u | \exists v \in S s.t. (u, v) \in E\}|$. For a probability distribution π over $[n]$, the *collision probability* is

$$||\pi||^2 = \sum_x \pi_x^2$$

   Show that the following is true:

- $||\pi||^2 \geq \frac{1}{|S(\pi)|}$ where $S(\pi) = \{x|\pi_x > 0\}$.
- $||\pi||^2 = ||\pi - u||^2 + \frac{1}{n}$ where u is the uniform distribution.
- If there is a constant $\lambda < 1$ such that the transition matrix of G is such that $|\lambda_2| \leq \lambda$ then for any $\alpha < 1$, G has vertex expansion $(\alpha n, \frac{1}{(1-\alpha)\lambda^2+\alpha})$.

Remark: We use the convention that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the random walk matrix.

5. (optional) Let $G = (V, E)$ be a d-regular, α-expander, i.e., $\forall S \subset V, |S| \leq |V|/2, |\lambda(S) \backslash S| \geq \alpha|S|$ (λ is defined as above). Let $n = |V|$. Suppose a set of pairs $S = \{(a_1, b_1), \ldots, (a_q, b_q)\}$ are such that a_i and b_i are chosen uniformly and independently from V for all i. Show that for all a, b, there is a path connecting a, b of length $O(\log n)$. Show that there exists a way of connecting each a_i, b_i pair via a path of length $O(\log n)$, such that no edge is used more than $O(\log n)$ times in total over all paths.

   Hint: for each (a_i, b_i) pair, choose a random x_i. Show how to pick a "random" path from a_i to x_i and from b_i to x_i. Show that your method is such that no edge is used more than $O(\log n)$ times in total.