6.842 Randomness and Computation	March 5, 2014
Homework 5	
Lecturer: Ronitt Rubinfeld	*Due Date: March 12, 2014*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. If you already knew the answer to one of the problems (call these "famous" problems), then let me know that in your solution writeup – it will not affect your score, but will help me in the future. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

The following problems are to be turned in. You should upload your solution to Stellar as a pdf file.

1. **(Edge expansion)** An n-vertex d-regular graph $G = (V, E)$ is called an (n, d, ρ)-edge expander if for every subset S of vertices satisfying $|S| \leq n/2$,

$$|E(S, \overline{S})| \geq \rho d |S|$$

where $E(S, T)$ denotes the set of edges $(u, v) \in E$ with $u \in S$ and $v \in T$.

Prove that for every n-vertex d-regular graph, there exists a subset S of $n/2$ vertices such that $E(S, \overline{S}) \leq dn/4$. Conclude that there does not exist an (n, d, ρ)-edge expander for $\rho > 1/2$.

2. **(Spectral expansion implies edge expansion)** Let G be a d-regular undirected graph on n vertices and with second largest eigenvalue $\lambda \leq 1$. Show that for any subset S of vertices of G,

$$|E(S, \overline{S})| \geq (1 - \lambda) \frac{d|S||\overline{S}|}{|S| + |\overline{S}|}$$

where $E(S, \overline{S})$ is the set of edges going between vertices in S and vertices in $\overline{S}$.

3. **(Random bipartite graphs are good vertex expanders)** A graph $G = (V, E)$ is called an (n, d, c)-vertex expander if it has n vertices, the maximum degree of a vertex is d and for every subset $W \subseteq V$ of cardinality $|W| \leq n/2$, the inequality $|N(W)| \geq c|W|$ holds, where $N(W)$ denotes the set of all vertices in $V \setminus W$ adjacent to some vertex in W. By considering a random bipartite 3-regular graph on $2n$ vertices obtained by picking 3 random permutations between the 2 color classes, prove that there exists $c > 0$ such that for every n there exists a $(2n, 3, c)$-vertex expander.