6.842 Randomness and Computation March 12, 2014

Homework 6

Lecturer: Ronitt Rubinfeld *Due Date: March 19, 2014*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. If you already knew the answer to one of the problems (call these "famous" problems), then let me know that in your solution writeup – it will not affect your score, but will help me in the future. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

The following problems are to be turned in. You should upload your solution to Stellar as a pdf file.

1. **(Universal Traversal Sequences)** We say that an undirected graph on n nodes is *labeled* if the edges adjacent to each vertex are labeled with numbers from 1 to n, and no two edges are labeled with the same number. An edge may be labeled differently on each of its endpoints.

   Given is a labeled graph G on n nodes, a node v in G, and a string $s = (s_1, \ldots, s_k) \in \{1, \ldots, n\}^k$, consider the following procedure. Our initial position is v. In the i-th step, if there is an edge adjacent to the current node, labeled with s_i, we follow that edge. Otherwise, we stay at the current node. We call s a (G, v)*-cover* if it can be used to visit all vertices of G by following to the above procedure.

   Let $\{1, \ldots, n\}^\star$ denote the set of all finite-length strings with elements in $\{1, \ldots, n\}$. A string $s \in \{1, \ldots, n\}^\star$ is a *universal traversal sequence* for size n if for every labeled connected graph G on n nodes and every node v in G, s is a (G, v)-cover.

   (a) Show that there exists a universal traversal sequence for size n of length $n^{O(1)}$.

   (b) (optional) Show that there exists a universal traversal sequence for size n of length $n^{O(\log n)}$ that can be constructed in $n^{O(\log n)}$ time.

2. **(Influence of variables on functions)** For $x = (x_1, \ldots, x_n) \in \{\pm 1\}^n$, let $x^{\oplus i}$ be x with the i-th bit flipped, that is,

   $$x^{\oplus i} = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

   The *influence of the i-th variable on* $f : \{\pm 1\}^n \to \{\pm 1\}$ is

   $$\mathrm{Inf}_i(f) = \Pr_x \left[f(x) \neq f\left(x^{\oplus i}\right) \right].$$

   The *total influence* of f is

   $$\mathrm{Inf}(f) = \sum_{i=1}^{n} \mathrm{Inf}_i(f).$$

   A function $f : \{\pm 1\}^n \to \{\pm 1\}$ is *monotone* if for all $x, y \in \{\pm 1\}^n$ such that $x_i \leq y_i$ for each i, $f(x) \leq f(y)$.

(a) Show that for any monotone function $f : \{\pm 1\}^n \to \{\pm 1\}$, the influence of the i^{th} variable is equal to the value of the Fourier coefficient of $\{i\}$, that is $\inf_i(f) = \hat{f}(\{i\})$.

(b) Show that the majority function $f(x) = \mathrm{sign}(\sum_i x_i)$ maximizes the total influence among n-variable monotone functions mapping $\{\pm 1\}^n$ to $\{\pm 1\}$, for n odd.

3. **(NAE test)** For function $f : \{\pm 1\}^n \to \{\pm 1\}$, the NAE test chooses $x, y, z \in \{\pm 1\}^n$ by choosing, independently for each i, the triple (x_i, y_i, z_i) uniformly from the set of "not all equal" triples (that is, all 3-tuples from $\{\pm 1\}$ except for $(1, 1, 1)$ and $(-1, -1, -1)$). Then, the test accepts iff the three outcomes $(f(x), f(y), f(z))$ are not all equal. Show that the probability that the NAE test passes a function f is

$$\frac{3}{4} - \frac{3}{4} \sum_{S \subseteq [n]} \left(\frac{-1}{3}\right)^{|S|} \hat{f}(S)^2$$