Lecture 18:

Local Computation Algorithms
for MIS
Maximal Independent Set:

\[ \text{def. } U \subseteq V \text{ is "Maximal Independent Set" (MIS)} \]

\[ \text{(1) } \forall u_1, u_2 \in U \quad (u_1, u_2) \notin E \quad \text{ (independent)} \]

\[ \text{(2) } \nexists w \notin U \text{ s.t. } U \cup \{w\} \text{ is also independent} \quad \text{(maximal)} \]

Today's assumption (important):

- \( G \) has max degree \( d \)

Note: Maximum Independent set is NP-complete

Maximal " " can be solved by greedy
Distributed Algorithm for Maximal Independent Set (MIS): "Luby's Algorithm" (one of many variants):

- all nodes set to "live"

- repeat $K$ times in parallel:

  - $V$ nodes $v$, $v$ "selects" self with prob $= \frac{1}{2d}$

  - if $V$ live: if $v$ selects self) $\iff$ (no nbr $w$ of $v$ selects itself)

  - then

    - (1) $v$ added to MIS
    - (2) $V +$ nbrs of $v$ removed from graph (set to "dead")

(for purposes of analyses, continue to select selves even after "die")

If goal is to "kill" the whole graph:

**Thm.** $\Pr \left[ \text{ # phases } \geq 8 \log n \right] \leq \frac{1}{n}$

**Corr.** $\mathbb{E}[\text{ # phases }]$ is $O(d \log n)$ $\Leftrightarrow$ can improve!!
Main Lemma \[ \Pr \{ \forall v \text{ live } \Rightarrow \text{ adds self to MIS in one round} \} \geq \frac{1}{yd} \]

**Pf.** \[ \forall v \text{ live } \]

\[ \Pr \{ v \text{ selects self} \} = \frac{1}{2d} \]

\[ \Pr \{ \text{ any } w \in N(v) \text{ selects self} \} \leq \sum_{w \in N(v)} \frac{1}{2d} \text{ union bound} \]

\[ \leq \frac{d}{2d} = \frac{1}{2} \]

\[ \therefore \Pr \{ \forall v \text{ selects self } \cap \text{ no nbr selects self} \} \geq \frac{1}{2d} \left( 1 - \frac{1}{2} \right) = \frac{1}{yd} \]

\[ \Rightarrow \text{ Corr. } \Pr \{ \forall v \text{ alive after } \frac{4}{y} kd \text{ rounds} \leq \left( 1 - \frac{1}{y_4} \right)^{\frac{4}{y} kd} \leq e^{-k} \]

Note: Luby's alg uses \( K = O(\log n) \)

so union bound \( \Rightarrow \) all die

(Also avoids dependence on \( d \) via smarter analysis)
Local Computation Algorithm for Luby's answer:

- Previous with $k = \Omega(d \log d)$ gives:

  $O(d \log d)$ round distributed alg outputting

  one of

  \begin{align*}
  \text{live} & - \nu \text{ alive after } \log d \text{ rounds} \\
  \text{in} & - \nu \text{ in MIS} \\
  \text{out} & - \nu \text{ not in MIS for sure}
  \end{align*}

- Using "Parnas Ron" reduction: on input $\nu$:

  Simulate $\nu$'s view of computation in $O(d \log d)$ queries to input

  Degree $d$ rounds

  Size of radius $O(\log d)$ ball around $\nu$

  Output whether $\nu$ is alive, in or out of MIS

  Subroutine Luby status($\nu$)

If $\nu$ is in/out, we are done!

What if $\nu$ is still alive?

To show:

can still figure $\nu$ out quickly

given subroutine Luby status
LCA for Mls(v):

if Luby-status(v) is in/out, output it & halt \(3d\) runtime

else (1) do BFS to find v’s connected component \(3d\times \text{size of component}\)

(2) Compute lexicographically 1st Mls \(H’\) to that connected component

(3) output whether v in/out of \(H’\)

• in
• out
• v’s live component

Runtime: need to bound size of live components
Bounding live component sizes:

\[ A_v = \begin{cases} 0 & \text{if } v \text{ survives all rounds} \\ 1 & \text{otherwise} \end{cases} \]

\[ B_v = \begin{cases} 1 & \text{if } \# \text{rounds s.t. } r \text{ picks self or no } w \in N(r) \text{ picks self} \\ 0 & \text{otherwise} \end{cases} \]

Claim if \( v \) survives, \# round s.t. \( v \) picks self or no \( w \in N(r) \) picks self

\[ \text{independent for } v \text{ and } \bar{v} \text{ at distance } 2 \]

Distance 2: Survival of both \( v \) and \( \bar{v} \) depends on whether \( w \) picked self

\( \Rightarrow \) not independent

Correction: \( A \) \( W \), if all nodes in \( W \) survive then

\[ \# \text{round s.t. any node } v \text{ in } W \text{ picks self or no } w \in N(v) \text{ picks self} \]

Note: (1) survival of \( v \) can depend only on coin tosses of \( W \)'s within distance 2 of \( v \)

\[ \Rightarrow \leq d^2 \text{ other } B_w \]'s \]
Note (2):
Survival of \( v \) is "rare" over \( c \cdot \log d \) rounds

\[
\Pr \left[ \text{round } k \text{ s.t. } v \text{ picks self \& no } w \in N(v) \text{ picks self} \right] 
\leq \left(1 - \frac{1}{4d}\right)^{c \cdot \log d}
\]

\[
\leq \frac{1}{8d^3}
\]

Notes (1) + (2) \implies
Survival "rare" + "\& independent"

\implies good behavior: surviving nodes in small connected components

\[\text{Surviving} \]

\[\text{NO} \]

\[\text{MAYBE} \]
Why can't we say anything about complete graphs?

Component of size $k$ survives

1. $\binom{n}{k}$ components $\leq n(4d^3)^k$ for deg-$d$ graphs
2. Survival within or without of component not independent
   $\uparrow$
lots of "dependencies"
Claim: After $O(\log d)$ rounds, connected components of survivors are of size $\leq O(\text{polylog} d \cdot \log n)$.

$\Rightarrow$ can use brute force!

Proof of Claim:

Idea:

1. Any connected component that is large has lots of nodes that are independent (distance $\geq 3$).
2. These independent nodes are unlikely to simultaneously survive.

Let $H^{(3)} \leftarrow$ graph such that nodes $\sim B_v$, edges $\sim B_v + B_w$ s.t. $v$ and $w$ are independent. Let $H^{(3)}$ be a distance = 3 in $G$.

$$\deg(H^{(3)}) \leq d^3$$

Observe: # connected components of size $w$ $\leq$ # size $w$ subtrees of $H^{(3)}$.

Why? Map each connected component $C$ to arbitrary spanning tree of $C$.

Mapping is 1-1 (note that each component could have many spanning trees).
Theorem: \( n \) size \( w \) trees in \( H^{(3)} \) \( \leq n \left( \frac{4d^3}{w} \right)^w \)

Why? 

\# nonisomorphic trees on \( w \) nodes \( \leq 4^w \)

Process:
- choose root
- choose tree
- choose placement within \( H^{(3)} \)

\( \# \) choices:
\( n \cdot 4^w \cdot (d^3)^w \)

Total \# choices:
\( n \cdot 4^w \cdot (d^3)^w \)

Note: Independent set in \( H^{(3)} \) \( \Rightarrow \) Is nodes dist 3 in \( G \) (pairwise)

\( \Rightarrow \) \( Pr[\text{ind set I survives in } G] \leq \left( \frac{1}{8d^3} \right)^{|I|} \)

\( \Rightarrow \) \( Pr[\text{specific size } w \text{ tree survives in } H^{(3)}] \leq \left( \frac{1}{8d^3} \right)^w \)

\( \Rightarrow \) \( Pr[\exists \text{ size } w \text{ tree surviving in } H^{(3)}] \leq n \left( \frac{4d^3}{w} \right)^w \)

\( \Rightarrow \) for \( w = \Theta(\log n) \),

\( Pr[\exists \text{ size } w \text{ tree surviving in } H^{(3)}] \leq \frac{1}{n} \leq \frac{1}{d^w} \)
\[ \Pr \left[ \text{exists size } w(d^3) \text{ component surviving in } G \right] \leq \frac{1}{n} \]

So, unlikely to have any surviving component of size \( \Omega(d^3 \log n) \).