Lecture 6:

- Testing
- Planarity
- Minor-Freeness

- Partition
- Oracles
Property Testing

All graphs

$\varepsilon$-close to $P$

Graphs with property $P$

Can we distinguish in sublinear time?

$P =$ "planar"

Compromise

Can we distinguish graphs with prop $P$ from far from $P$?

e.g. $G$ is $\varepsilon$-far from planar if must remove $\geq \varepsilon \cdot d_{\text{max}} n$ edges to make it planar

Today: Test planarity in time independent of $n$

(but exponential in $\varepsilon$)
Testing H-minor freeness

all graphs have max degree ≤d

def. H is "minor" of G if can obtain H from G via vertex removals, edge removals, edge contractions

G is "H-minor-free" if H not minor of G

G is "ε-close to H-minor-free" if can remove ≤εdεn edges to make it H-minor-free

orb.: G is "ε-far"

• minor closed property P:
  if G ∈ P then all minors of G are in P

Really Cool Theorem [Robertson + Seymour]

Every minor-closed property is expressible as a constant k of excluded minors.

Some minor-closed properties: \( K_{3,3} \) or \( K_5 \), planar graphs, bounded treewidth, ...

Goal: Testing H-minor freeness
Pass H-minor free graphs
Fail if far from H-minor-free
more definitions

\( x \) can be a function of \( E \)

- \( G \) is "\((E, k)\) - hyperfinite" if
  - Can remove \( \leq E n \) edges
  - Remain with connected components of size \( \leq k \)

(i.e., can remove few edges and break up graph into very small components.)

Useful Thm

Given \( H \) constant that depends only on \( H \)

exists \( C_H \) s.t. \( 0 < C_H \leq 1 \), every \( H \)-minor free graph of \( \text{deg} \leq d \)

is \( (Ed, C_H / C_H) \)-hyperfinite.

(i.e., remove \( \leq Edn \) edges \& components of size \( O(1/C_H) \))

\( \leq E \) fraction

\( \leq E \) fraction

note

Subgraphs of \( H \)-minor free graphs also \( H \)-minor free

\( \Rightarrow \) so also hyperfinite

but only remove \#edges in proportion to \#nodes in subgraph

\( \Rightarrow \) Can "recurse" \& break up further

hyperfinite graphs

\( H \)-minor free graphs
Why is hyperfiniteness useful?

Partition graph $G$ into $G'$
- only constant-size connected components remain
- removed only few edges ($\leq Ed_n$)
  - if can't do this, $G$ is not $H$-minor-free

If $G'$ is close to having property, so is $G$

Constant time
- so test $G'$ by picking random components & seeing if they have the property

Need a "local" (sublinear) way to determine $G'$: For now assume we have "partition oracle" $P$
(with parameters $\frac{Ed_n}{\eta}$, $k$)
  - component size fraction edges removed

Input: vertex $v$
Output: $P[v]$ (v's partition name)

s.t. \forall v \in V
  \begin{align*}
  (1) \ &|P[v]| \leq k \\
  (2) \ &P(v) \text{ connected}
  \end{align*}

If $G$ is $H$-minor-free
(with prob $\geq \frac{9}{10}$)

$$|\{u \in V \mid \exists (u,v) \in E \mid P(u) \neq P(v)\}| \leq \frac{Ed_n}{4}$$

Easy to test since collection of constant sized graphs!!
Algorithm given partition oracle \( P \):

I. Does partition oracle give partition that "looks right"? e.g. few crossing edges

1. \( A \leftarrow \text{estimate of \# of edges } (u,v) \)
   s.t. \( P(u) \neq P(v) \) to additive error \( \leq \frac{edn}{8} \) with prob of failure \( \leq \frac{1}{10} \)

   - if \( \frac{A}{\frac{3}{8} Edn} \geq \) output "fail" + halt

II. Test random partitions

   - Choose \( S' = O(\frac{1}{\epsilon^2}) \) random nodes \( S \) select "random" partitions

   - if for any \( s \in S \), \( P(s) \geq k \) or \( P(S) \) not \( H \)-minor free, reject + halt

   - size \( k = O(\frac{1}{\epsilon^2}) \)

   - so easy to test

   - Accept

Runtime:

Part I: \( O(\frac{1}{\epsilon^3}) \) calls to oracle

Part II: \( O(\frac{d}{\epsilon^2}) \) calls to oracle to determine \( P(S) \)

\( O(\frac{d}{\epsilon^2}) \) total calls
Analysis (assume oracle P always correct)

* if $G$ is $H$-minor free:

1) $E[\hat{F}] \leq \frac{Edn}{4}$

Sampling bounds (Chernoff/Hoeffding) $\Rightarrow \hat{F} \leq \frac{Edn}{4} + \frac{Edn}{8} = \frac{3}{8} Edn \Rightarrow$ algorithm doesn't fail at stage I with prob $\geq \frac{9}{10}$

2) $\forall S \subseteq V, P[S]$ is $H$-minor free

* if $G$ is $\epsilon$-far from $H$-minor free:

Case 1: $P$'s output doesn't satisfy $|\{ (u, v) \in E : P(u) \neq P(v) \}| \leq \frac{Edn}{2}$

Sampling bounds $\Rightarrow \hat{F} \geq \frac{Edn}{2} - \frac{Edn}{8} = \frac{3}{8} Edn$

$\Rightarrow$ output "fail" with prob $\geq \frac{9}{10}$

Case 2: $P$ satisfies $|\{ (u, v) \in E : P(u) \neq P(v) \}| \leq \frac{Edn}{2}$

$G' \leftarrow G$ with edges in $C$ removed

Note: $G'$ is $\frac{\epsilon}{2}$-close to $G$

so, if $G$ is $\epsilon$-far from having property, then $G'$ is $\frac{\epsilon}{2}$-far from having property!
Since $G'$ is $\frac{\varepsilon}{2}$-far from $H$-minor free, it must change $\geq \frac{\varepsilon d n}{2}$ edges, which touch $\geq \frac{\varepsilon n}{2}$ nodes. So, with prob $\geq \frac{\varepsilon}{2}$, pick a node in a component which is not $H$-minor free.

**Remaining Issue:**
Implementing partition oracle $P$

**Plan:**
1) Define global partitioning strategy (not sublinear time)

2) Figure out how to implement locally (only find partition of given node, not whole solution)
A useful concept—
"Isolated Neighborhoods"

def $S$ is "$(\delta, k)$-isolated neighborhood of node $v$":

1) $v \in S$
2) $S$ connected
3) $|S| \leq k$
4) # edges connecting $S + \overline{S} \leq \delta |S|$

In hyperfinite graphs, most nodes have $(\delta, k)$-isolated neighborhoods.

Is this obvious?

- $G$ hyperfinite $\Rightarrow \exists$ partitioning
- but will need this to be true about remaining graph in context of algorithm that may find a different partition "step-by-step"

- Luckily, no matter what was removed earlier, we still have an $H$-minor-free graph so still hyperfinite!
Global Partitioning Algorithm \rightleftharpoons a "mental thought process"

Let \( \Pi_1 \ldots \Pi_n \) be nodes in random order

\( P \leftarrow \emptyset \)

For \( i = 1 \ldots n \) do

- if \( \Pi_i \) still in graph then
  - if \( \exists (S, k) \)-isolated nbhd of \( \Pi_i \) in remaining graph
    - then \( S \leftarrow \) this nbhd
  - else \( S \leftarrow \exists \Pi_i \S \)

\( P \leftarrow P \cup \exists S \S \)

Remove \( S \) + adjacent edges from graph

Does this give a partition with few crossing edges?

- \( S \) s.t. \( S \) is \((8, k)\)-isolated contribute \( \leq 8|S|\) edges which overall \( \leq 8 \cdot n \)

- \( S \) s.t. \( S = \exists \Pi_i \S \) (one node):
  - need to show that not too many of these!

\[\delta = \frac{Ed}{n}, \quad k = \left( \frac{A}{\varepsilon^2} \right)\]
Lemma: if $G'$ is subgraph of a (hyperfinite) graph $G$ s.t. $G'$ has $\geq 8n$ nodes

then \( \leq \frac{\varepsilon}{30} \) fraction of nodes in $G'$ don't have $(\delta, k)$-isolated nbhds, for $\delta = \frac{\varepsilon}{30}$ and $k = \Theta(\varepsilon^3)$

Proof idea:

$G$ is minor free $\downarrow$

$G'$ is minor free $\downarrow$

$G'$ is hyperfinite $\downarrow$

exists partition s.t. most nodes in $G'$ are in $(k, \delta)$-isolated nbhd $\downarrow$

$T_i$ randomly chosen in $G'$ $\downarrow$

whp $T_i$ in $(k, \delta)$-isolated nbhd.

So, not too many "singletons"!"
Local Simulation of Partitioning Oracle:

- Input \( v \)
- Assume access to random function \( \Pi(v) \)
  \[ \Pi : v \rightarrow [n] \]
- Output \( P[v] \)

- Recursively compute \( P[w] \) for all \( w \) s.t.
  \[ \Pi(w) < \Pi(v) \]
  \[ w \text{ is distance } \leq 2k \text{ from } v \]

- If \( \exists w \text{ s.t. } v \in P[w] \)
  then \( P[v] = P[w] \)
  else look for \((k, \delta)\)-isolated nbhd of \( v \)
    (ignoring nodes which are in \( P[w] \) for smaller ranked \( w \)'s)
    if find one, \( P[v] \leftarrow \text{this nbhd.} \)
    else \( P[v] \leftarrow \emptyset \)

Query Complexity:

\[ d^{\Theta(k)} \]

using analysis from last time + \( k \times \Theta(\varepsilon^3) \)

but can do much better:

Currently \( d^{O(\log^3(1/\varepsilon))} \) is possible.