Today:

Undirected $S-T$ Connectivity revisited
(deterministic logspace)

**Given:** undir $G$

nodes $s, t$

**Question:** are $s, t$ in same component?

an easy case:

**def.** $(N, D, \lambda)$-graph

$\uparrow$ $\uparrow$

$\# $nodes degree upper bd of transition matrix

a well known fact: 
Tanner, Allen-Millman

$\forall \lambda < 1, \exists \varepsilon > 0 $ s.t. $\forall (N, D, \lambda)$-graph

$+ A S $ s.t. $15 \leq N \over 2 \quad |N(S)| \geq (1+\varepsilon)|S|$

includes $S$
\[ \lambda < 1 \implies G \text{ has } O(\log n) \text{ diameter} \]

Idea for "low diameter + const degree"

- (each component is low diameter)
- starting at \( s \):
  - enumerate all paths of length \( O(\log n) = l \)
    - \( \# \text{ paths} = N = O(l) = O(\log n) = N \)
      - since \( D = O(1) \)
  - if ever see \( t \), output "connected" (\( \text{O.w.} \) "disconnected")

Correct? \( \checkmark \)

Space: Keep track of DFS
- const \# bits for each step
- \( O(\log n) \) length
- Total \( O(\log^2 n) \)
Problem: not all graphs \((N,0,\lambda)\) for \(\lambda < 1\)

- \(O(\log N)\) diam
- Const deg

\(O((\log n)^2)\) Solution:

- Keep track of nodes on DFS stack
- \(S = 1\)

- \(L\) \(L\) \(L\)
- \(L\) \(L\) \(R\)
- \(L\) \(R\) \(L\)

\(O(\log n)\) Solution:

- Keep track of choices on DFS stack
- Can find parents by looking for start of choices
- \((\log n \cdot O(1) + O(1))\) to find previous starting at root and following choices
For general graphs:

Thm 4 connected, non-bipartite \( \lambda(S) \leq 1 - \frac{1}{\text{DN}_2} \)

What about powering?

\( G \) is \((N, D, \lambda)\) \( \rightarrow \) \( G^t \) is \((N, D^t, \lambda^t)\)

good/bad?

+ same soln
+ reduce \( \lambda_2 \)
- increased degree

will power but will add operation
which reduces degree
w/o increasing \( \lambda_2 \) by too much
"Base graph"

Thm 1  \[ \exists \text{ const } D_e \geq (|D_e|^{16}, D_e^{1/2})\text{-graph} \]

- const size graph that has small \( \lambda_2 \)
- can use it for any input
- can find via enumeration

Can we assume \( G \) is const degree?

Transform \( G \)

\[ G': \begin{align*}
\text{degree } \leq N \\
\text{# nodes } = N
\end{align*} \]

\[ \Rightarrow \text{ new } G \]

\[ \begin{align*}
\text{degree } \leq 3 \\
\text{# nodes } \leq N^2
\end{align*} \]

Same connectivity properties
Representing graphs:

Rotation map: \( \text{Rot}_6 : [N] \times [N] \rightarrow [N] \times [N] \)

\( \text{Rot}_6 (v, i) = (w, j) \) if

- \( i \)th edge of \( v \) leads to \( w \)
- \( j \)th edge of \( w \) leads to \( v \)

allows back & forth on same edge

\[ G: \]

\[ x \quad \bullet \quad 0 \quad \bullet \quad 0 \quad \bullet \quad u \quad v \quad w \]

Replacement Product \( G \odot H \)

Given \( G, d\text{-reg}, N \) nodes \( \Rightarrow G' \) \( N \cdot D \) nodes

\( H, d\text{-reg}, D \) nodes \( \Rightarrow \)

reduces degree, what does it do to \( \lambda \)?
nodes: \( v \in G \) replaced by copy \( H \)

edges: each vertex in \( H_v \) connected to nbors in \( H_v \)

- if \( u \) is \( i \)th nbr of \( v \) in \( G \)
  - \( v \) is \( j \)th nbr of \( u \)
  - add edge from \( i \)th node of \( H_v \)
    to \( j \)th " " \( H_v \)

\[ G \]
\[ x \quad 2 \quad u \quad v \]
\[ H \]
\[ 1 \quad 2 \quad 3 \]

\[ \Rightarrow G \bowtie H \]

\[ x \quad \cdots \quad u \quad \cdots \quad v \quad \cdots \quad w \cdots \]
Zig Zag Product $G@H$

Given $G$ D-reg $N$ nodes $\Rightarrow G''$ with $N\cdot D$ nodes $H$ d-reg $D$ nodes $\Rightarrow d^{2}$

nodes: as in $G'$
  each $v \in G$ replaced by copy of $H$

edges: path of length 3 in $G'$
  $(u,v) \in G''$ iff $u \in H_{i}$ "cloud i"
    $\exists w \in H_{i}$ s.t. $(w,v) \in E(H_{i})$
  $(w,z) \in G@H$
  $(z,v) \in E(H_{j})$ where $v \in H_{j}$

new degree $d^{2}$
Thm: For $\alpha \leq \frac{1}{2}$

$G$ an $(N,0,\lambda)$-graph and $H$ a $(N,d,\lambda)$-graph

$G \boxtimes H$ is $(N,0,d,\lambda_{G \boxtimes H})$-graph

s.t.

$\frac{1}{\alpha} (1-\alpha^2) (1-\lambda) \leq 1 - \lambda_{G \boxtimes H}$

So

$\lambda_{G \boxtimes H} \leq 1 - \frac{1}{2} \left(1-\alpha^2\right)(1-\lambda)$

$\leq 1 - \frac{3}{8} \lambda$ ($\leq \frac{2}{3} + \frac{\lambda}{3}$) - still $< 1$

So degree drops + $\lambda_2$ isn't so bad

How to use?

Main transformation:

Given:

$G$ $D^{16}$-reg on $N$ nodes
$H$ $D$-reg on $D^{16}$ nodes
Transformation:

\[ \lambda = \text{smallest int s.t. } \left( 1 - \frac{1}{DN^2} \right)^d \leq \frac{1}{2} \]

\[ G_0 \leftarrow G \]

\[ G_{\lambda} \leftarrow (G_{\lambda-1} \circ 
\text{deg reduction} \right)^8 \]

Output: \[ G_\lambda \]

Properties of \( G_{\lambda} \):

\# nodes = \( N (DN^4)^{\ell} \)

degree is \( O(1) \)

= \( \text{poly}(N) \)

Lemma: \( \lambda(G_\lambda) \leq \frac{1}{2} \) so diameter is small

Use alg in beginning of class

on \( G_\lambda \)