Today:

Weak learning of monotone functions.

def partial order \leq

\[ x \leq y \quad \text{iff} \quad x_i \leq y_i \]

\begin{align*}
0011010 & \leq 0111011 \\
\neq 0001111 & \quad f(x) \neq f(y)
\end{align*}

Monotone function f

\[ x \leq y \Rightarrow f(x) \leq f(y) \]

Can you learn the class of monotone functions?

In h.w. 2a, random samples suffice for uniform distribution.
Arbitrary distributions hard, even with queries

\[ \text{Occam} \Rightarrow \# \text{monotone} \text{FCT} \approx 2^{2^{2^n}} \]

\[ \text{so} \quad \frac{2^n}{n} \text{ samples suffice} \]

Consider "Slice FCT"

\[ \sum \#1's > \#0's \quad \text{row where} \quad \#1's = \#0's \]

\[ \sum \#0's > \#1's \]

\[ \text{can be anything} \]

\[ \approx \frac{2^n}{n} \text{ elts in middle row} \]

\[ 2^n \text{ possible} \]

\[ \text{a "hard" distribution,} \]

uniform on middle row

any learning alg needs to see
most of middle row (purple)

\[ \Rightarrow \quad o \left( 2^{2^n} \right) \text{ queries/samples} \]

in PAC model

\[ \text{Today, uniform on } 2^{0.13^n} \text{ (whole hypercube)} \]

with queries

can get slight win!
Can weakly learn:

all monotone ftsns have weak agreement with some dictator ftns,

∃ X₁, X₂, ..., Xₙ s.t. f(x) = x_i

(switch to f: ±1^n → ±1)

Thm: If monotone, ∃ g ∈ {±1, X₁, X₂, ..., Xₙ} s.t. Pr_x[ f(x) = g(x) ] ≥ 1/2 + Ω(1/n)

If true ⇒ gives alg for weak "learning" of f with agreement 1/2 + Ω(1/n)

by testing all ftns in S

and outputting any one that agrees

≥ 1/2 + Ω(1/n)

Pf of Thm

Case 1: f(x) has 3/4-agreement

with +1 or -1
Case 3: \( \Pr [f(x) = 1] \in \left[ \frac{1}{4}, \frac{3}{4} \right] \)

"balanced"

First a "break":

**Define "Influence"**

- # nodes = \( 2^n \)
- # edges = \( \frac{n \cdot 2^n}{2} \)
- Each level has \( \binom{n}{j} \) with \( j \) nodes
- Monotone \( \Rightarrow \) no blue above any red

\[
\tilde{I}(f) = \frac{\# \text{red-blue edges}}{2^{\alpha-1}} = \Pr_x [f(x) \neq f(x^{(i)})]
\]

\[
\tilde{I}_i(f) = \frac{\# \text{red-blue edges in } i^{th} \text{ direction}}{2^{\alpha-1}}
\]

On h.w.: for monotone \( f \)

\[
\text{Thm: } \tilde{I}_i(f) = f(x_i^2) \equiv 2 \cdot \Pr[f(x_i^2) = 1]
\]

\[
X_{\Xi}^2 = \prod_{j \in \Xi^2} X_j = X_i
\]
\[
\text{Plan} \quad \text{Show } \exists i \inf_{i} (f) \geq A \left( \frac{1}{n} \right) \\
\Rightarrow \quad \Pr \left[ f(\omega) = X_i \right] \geq \frac{1}{2} + \frac{\inf_{i} (f)}{2} \\
\geq \frac{1}{2} + O \left( \frac{1}{n} \right)
\]

Important tool:

**Canonical Path Argument**

\[
\text{Plan} \\
1) \text{ define canonical path for every red-blue pair of nodes} \\
    \quad -(\# \text{ red nodes}) \times (\# \text{ blue nodes}) \text{ such paths} \\
    \quad \text{must cross at least one red-blue edge} \\
2) \text{ show upper bound on } \# \text{ of c.p.s} \\
    \quad \text{passing thru any edge} \\
3) \text{ conclude lower bound on } \# \text{ of red-blue edges}
\]
Part 1:

∀ (x,y) s.t. x is red (but not necessarily y is blue; x ≤ y or y ≤ x)

"canonical path" from x to y is:

- scan bits left to r, flipping where needed
- each flip ➔ step in path

Example:

\[
\begin{array}{ccccccc}
\text{X} & 1 & 2 & 3 & 4 \\
\text{w} & +1 & +1 & +1 & +1 \\
\text{z} & +1 & -1 & +1 & -1 \\
\text{y} & -1 & -1 & +1 & -1 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
x \uparrow \downarrow \downarrow \\
w \downarrow \\
z \downarrow \\
y \\
\end{array}
\]

Note: path can go up/down as much as it wants
how many red-blue x,y pairs?

Pr[f(x)=1] ∈ \([\frac{1}{4}, \frac{3}{4}]\)

\# paths ≥ \(\frac{1}{4} \cdot 2^n\) · \(\frac{1}{4} \cdot 2^n = \frac{1}{16} \cdot 2^n\)

1.b. on 1.b. on
\# red \# blue

Part 2 of plan:

For any (red-blue) edge e,

how many x-y pairs can cross it with canonical x-y path?

\(x\)

\[\ldots\ y_i \ldots x_i \ldots y_n\]

\(e=(u, u^{(i)}) \in \Theta\)

\(y_i \ldots y_i, x_i \ldots x_i \ldots y_n\)

\(y_i \ldots y_i \ldots y_i, x_i \ldots x_i \ldots x_n\)

\(y\)

\[\ldots\ y_i \ldots x_i \ldots x_n\]

\(\leq (2^{i-1}) \cdot 2^{n-i} 2^n\) total settings of prefix of x & suffix of y consistent with e

\(2^{i-1}\) possible x’s could reach here

\(2^{n-i}\) possible y’s could be reached
Part 3 of plan

$$(\text{\# red-blue edges}) \times (\text{max \# canonical paths that can use it})$$

$$\geq \text{\# red-blue canonical paths} \geq \frac{1}{16} \cdot 2^n$$

since each red-blue c.p. used a red-blue edge

So $\text{\# red-blue edges} \geq \frac{1}{16} \cdot 2^n = \frac{1}{16} \cdot 2^n$

$$= a(2^n)$$

So $\exists i \neq i' \geq 2^n \cdot \frac{1}{16} \cdot \frac{1}{n}$ red-blue edges in dir $i$
\[ \inf_x f(x) \geq \frac{2^n}{16} \cdot \frac{1}{n} = \frac{1}{8n} = f(\exists x \exists \hat{x}) \\
= 2 \cdot \Pr(f(x) = x) - 1 \]

\[ \Rightarrow \Pr(f(x) = x) \geq \frac{1}{2} + \frac{1}{16 \cdot n} \]