On Semi-implicit Splitting Schemes for the Beltrami Color Flow

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Abstract. The Beltrami flow is an efficient non-linear filter, that was shown to be effective for color image processing. The corresponding anisotropic diffusion operator strongly couples the spectral components. Usually, this flow is implemented by explicit schemes, that are stable only for small time steps and therefore require many iterations. In this paper we introduce a semi-implicit scheme based on the locally one-dimensional (LOD) and additive operator splitting (AOS) schemes for implementing the anisotropic Beltrami operator. The mixed spatial derivatives are treated explicitly, while the non-mixed derivatives are approximated in a semi-implicit manner. Numerical experiments demonstrate the stability of the proposed scheme. Accuracy and efficiency of the splitting schemes are tested in applications such as the scale-space analysis and denoising. In order to further accelerate the convergence of the numerical scheme, the reduced rank extrapolation (RRE) vector extrapolation technique is employed.

1 Introduction

Nonlinear diffusion filters based on partial differential equations (PDEs) have been extensively used in the last decade for different tasks in image processing. Their efficient implementation requires designing numerical schemes in which the issues of accuracy, stability, and computational cost all play important roles.

The Beltrami image flow is an example of a non-linear filter, that is efficient for color image processing. It treats the image as a 2-D manifold embedded in a hybrid spatial-feature space. Minimization of the image area surface yields the Beltrami flow. The corresponding diffusion operator is anisotropic and strongly couples the spectral components. Due to its anisotropy and non-separability, so far there is no efficient implicit, nor operator-splitting-based numerical scheme for the partial differential equation that describes the Beltrami flow in color. Usual discretizations of this filter are based on explicit schemes, that limit the time step and therefore result in a large number of iterations. In [1] an acceleration technique based on the reduced rank extrapolation (RRE) algorithm [2, 3] was proposed in order to speed-up the slow convergence of the explicit scheme.

As an alternative to the explicit scheme, an approximation using the short time kernel of the Beltrami operator was suggested in [4]. Although unconditionally stable, this method is still computationally demanding, since computing the kernel involves geodesic distance computation around each pixel.

The bilateral filter, which can be shown to be an Euclidean approximation of the Beltrami kernel, was studied in different contexts (see [5], [6], [7], [8], [9], [10]). Recently, a related filter, the nonlocal means filter, was proposed in [11] and shown to be useful in denoising gray-scale and color images.

In this paper we propose to approximate the system of nonlinear coupled equations given by the Beltrami flow using a semi-implicit finite difference scheme based on operator splitting. Historically, additive operator splitting (AOS) schemes were first developed for (nonlinear elliptic/parabolic) monotone equations and Navier-Stokes equations [12, 13]. In image processing applications, the AOS scheme was found to be an efficient way for approximating the Perona-Malik filter [14], especially if symmetry in scale-space is required. The AOS scheme is first order in time, semi-implicit, and unconditionally stable with respect to its time-step [13, 14]. In the early 1950's (see [15]) the alternating-direction method (ADI) was introduced, and in [16] the LOD (locally one-dimensional) splitting method was proposed. The LOD scheme and other multiplicative splitting methods were employed in the context of nonlinear diffusion image filtering in [17]. We stress that the main characteristic of this class of equations, which allows splitting, is local isotropy. However, in the case of the anisotropic Beltrami operator, the main difficulty in splitting stems from the presence of the mixed derivatives. To overcome this problem, we suggest to construct the following semi-implicit scheme; the spatial mixed derivatives are discretized explicitly at the current time step $n\Delta t$, while those that do not contain mixed derivatives are approximated using an average of two levels of time steps: $n\Delta t$ and $(n+1)\Delta t$ (Crank-Nicolson scheme). As our equations are nonlinear, a stability proof of the corresponding finite difference scheme is a non-trivial task. We provide numerical experiments which indicate that the LOD and the AOS splitting schemes for the nonlinear Beltrami color filter are stable for a wide range of time steps. We demonstrate the efficiency and stability of the splitting in applications such as: Beltrami-based scale space and Beltrami-based denoising. In order to further expedite the LOD/AOS splitting schemes, we show how to speed-up their convergence by using the RRE (reduced rank extrapolation) technique. The RRE method was introduced by Mešina and Eddy [2, 3] to speed-up the convergence of general sequences of vectors without explicit knowledge of the sequence generator. This technique was applied in [1] in order to speed up the slow convergence of the standard explicit scheme for the Beltrami color flow. In this paper we show that in applications such as scale-space and denoising of color images, the semi-implicit LOD/AOS schemes can also be accelerated using the RRE technique.

This paper is organized as follows: In Section 2 we briefly summarize the Beltrami framework. In Section 3 we briefly review general semi-implicit splitting operator schemes. In Section 4 we propose a semi-implicit splitting scheme for the anisotropic Beltrami operator, based on the LOD/AOS schemes. In Section 5 we demonstrate the efficiency and stability of the LOD/AOS splitting schemes for Beltrami-based scale-space and Beltrami-based denoising. Furthermore, we propose to accelerate the LOD/AOS schemes using the RRE technique. Section 6 concludes the paper.

2 The Beltrami Framework

Let us briefly review the Beltrami framework for non-linear diffusion in computer vision [18–21]. We represent images as embedding maps of a Riemannian manifold in a higher dimensional space. We denote the map by $U : \Sigma \to M$, where Σ is a two-dimensional surface, with (σ^1, σ^2) denoting coordinates on it. M is the spatial-feature manifold, embedded in $\mathbb{R}^{d+2}$, where d is the number of image channels. For example, a gray-level image can be represented as a 2D surface embedded in $\mathbb{R}^3$. The map U in this case is $U(\sigma^1, \sigma^2) = (\sigma^1, \sigma^2, I(\sigma^1, \sigma^2))$, where I is the image intensity. For color images, U is given by $U(\sigma^1, \sigma^2) = (\sigma^1, \sigma^2, I^1(\sigma^1, \sigma^2), I^2(\sigma^1, \sigma^2), I^3(\sigma^1, \sigma^2))$, where I^1, I^2, I^3 are the three components of the color vector.

Next, we choose a Riemannian metric on this surface, g, with elements denoted by g_{ij}. The canonical choice of coordinates in image processing is Cartesian (we denote them here by x^1 and x^2). For such a choice, which we follow in the rest of the paper, we identify $\sigma^1 = x^1$ and $\sigma^2 = x^2$. In this case, σ^1 and σ^2 are the image coordinates. We denote the elements of the inverse of the metric by superscripts g^{ij}, and the determinant by $g = \det(g_{ij})$. Once images are defined as embedding of Riemannian manifolds, it is natural to look for a measure on this space of embedding maps.

Denote by (Σ, g) the image manifold and its metric, and by (M, h) the space-feature manifold and its metric. Then, the functional $S[U]$ assigns a real number to a map $U : \Sigma \to M$,

$$S[U, g_{ij}, h_{ab}] = \int d^s\sigma \sqrt{g} ||dU||^2_{g,h}, \tag{1}$$

where s is the dimension of Σ, g is the determinant of the image metric, and the range of indices is $i, j = 1, 2, ... \dim(\Sigma)$ and $a, b = 1, 2, ... \dim(M)$. The integrand $||dU||^2_{g,h}$ is expressed in a local coordinate system by $||dU||^2_{g,h} = (\partial_{x_i} U^a) g^{ij} (\partial_{x_j} U^b) h_{ab}$. This functional, for $\dim(\Sigma) = 2$ and $h_{ab} = \delta_{ab}$, was first proposed by Polyakov [22] in the context of high energy physics, in the theory known as *string theory.* The elements of the induced metric for color images with Cartesian color coordinates are

$$G = (g_{ij}) = \begin{pmatrix} 1 + \beta^2 \sum_{a=1}^{3} (U^a_{x_1})^2 & \beta^2 \sum_{a=1}^{3} U^a_{x_1} U^a_{x_2} \\ \beta^2 \sum_{a=1}^{3} U^a_{x_1} U^a_{x_2} & 1 + \beta^2 \sum_{a=1}^{3} (U^a_{x_2})^2 \end{pmatrix}, \tag{2}$$

where a subscript of U denotes a partial derivative and the parameter $\beta > 0$ determines the ratio between the spatial and spectral (color) distances. Using standard methods in calculus of variations, the Euler-Lagrange equations with respect to the embedding (assuming Euclidean embedding space) are

$$0 = -\frac{1}{\sqrt{g}} h^{ab} \frac{\delta S}{\delta U^b} = \underbrace{\frac{1}{\sqrt{g}} \mathrm{div}\,(D \nabla U^a)}_{\Delta_g U^a}, \tag{3}$$

where the diffusion matrix is $D = \sqrt{g} G^{-1}$. Note that we can write

$$\mathrm{div}(D\nabla U) = \sum_{q,r=1}^{2} \partial_{x_q} (d_{qr} \partial_{x_r} U).$$

The operator that acts on U is the natural generalization of the Laplacian from flat spaces to manifolds. It is called the Laplace-Beltrami operator, and denoted by Δ_g.

The parameter β, in the elements of the metric g_{ij}, determines the nature of the flow. At the limits, where $\beta \to 0$ and $\beta \to \infty$, we obtain respectively a linear diffusion flow and a nonlinear flow, akin to the TV flow [23] for the case of grey-level images (see [20] for details).

The Beltrami scale-space emerges as a gradient descent minimization process

$$U^a_t = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta U^a} = \Delta_g U^a, \quad a = 1, 2, 3. \tag{4}$$

For Euclidean embedding, the functional in Eq. (1) reduces to

$$S(U) = \int \sqrt{g}\, dx^1\, dx^2. \tag{5}$$

This geometric measure can be used as a regularization term for color image processing. In the variational framework, the reconstructed image is the minimizer of a cost-functional. This functional can be written in the following general form,

$$\Psi(U) = \lambda \sum_{a=1}^{3} ||U^a - F^a||^2 + S(U),$$

where the parameter λ controls the smoothness of the solution and F is the given image.

The modified Euler-Lagrange equations as a gradient descent process are

$$U^a_t = -\frac{1}{\sqrt{g}} \frac{\delta \Psi}{\delta U^a} = -\frac{2\lambda}{\sqrt{g}} (U^a - F^a) + \Delta_g U^a, \quad a = 1, 2, 3. \tag{6}$$

3 Operator splitting schemes

In this section we briefly review standard first order accurate splitting schemes for diffusion equations. One of the main drawbacks of the semi-implicit schemes for such equations in multiple dimensions is that the resulting inverted matrix does not have an efficient algorithm for its inversion. In order to remedy this shortcoming, splitting techniques are commonly employed in solving time-dependent partial differential equations. They allow one to reduce problems in multiple spatial dimensions to a sequence of problems in one dimension, which are easier to solve.

One of the simplest splitting schemes belonging to the class of *multiplicative operator splitting schemes*, is the locally one-dimensional (LOD) scheme [16]. The LOD scheme only needs to invert one three-diagonal matrix for each direction. It is simple to implement, is unconditionally stable and it is first order accurate. However, the system matrix is not axis symmetric, a property that may be important in some cases. If such a property is required, one could use the additive operator splitting scheme [13], which was actually invented for parallel implementation of splitting methods.

Even for sequential implementations, the AOS is almost as efficient as the LOD scheme; instead of multiplying the operators, one computes them independently and then averages the sums of the inverse of the two matrices. We want to emphasize that the matrices for AOS use $2\Delta t$ instead of Δt.

It is not a trivial matter to apply dimensional splitting schemes for Beltrami type of equations. Our goal is to construct a splitting scheme for the nonlinear anisotropic Beltrami operator, which would amount to inverting tridiagonal matrices, be unconditionally stable and preserve the time discretization accuracy that was obtained without applying splitting techniques.

4 The proposed splitting scheme

In this section we present an operator splitting scheme for the Beltrami filter. Before splitting, we first introduce a semi-implicit approximation scheme to our equations. A semi-implicit Crank-Nicolson scheme for an equation involving mixed derivatives can rely on the following discretization of the spatial derivatives operators: mixed derivatives are computed at time step $n\Delta t$, while the non-mixed derivatives are computed as the average of the values at time steps $n\Delta t$ and $(n+1)\Delta t$. This approach for handling mixed derivatives in semi-implicit schemes for approximating linear equations has been considered in several previous works (see [24–26] for example), including the context of image processing [27], although it was not combined with the Crank-Nicolson method in the latter case. We note that in numerical experiments we have found the introduction of the Crank-Nicolson method into the splitting scheme necessary in order to maintain stability for large time steps. A simpler scheme, similar to the one used in [27], did not seem to be sufficiently stable for this PDE and the applications demonstrated in this paper. We now present the scheme we intend to use.

First, we refine our grid notations. We work on the rectangle $\Omega = (0,1) \times (0,1)$, which we discretize by a uniform grid of $m \times m$ pixels, such that $x_i = i\Delta x,\ y_j = j\Delta y,\ t_n = n\Delta t$, where $1 \le i \le m$, $1 \le j \le m$, $1 \le n \le J$ and $J\Delta t = T$. Let the grid size be $\Delta x = \Delta y = \frac{1}{m-1}$.

For each channel U^a, $a = 1,2,3$ of the color vector, we define the discrete approximation $(U^a)^n_{ij}$ by

$$(U^a)(i\Delta x, j\Delta y, n\Delta t) = (U^a)^n_{ij} \approx U^a(i\Delta x, j\Delta y, n\Delta t).$$

We impose von-Neumann boundary condition, and initially set U^a to be our initial data image.

4.1 LOD/AOS scheme for the Beltrami scale-space

We approximate the Beltrami filter given in Eq. (4) by the following semi - implicit Crank-Nicolson scheme:

$$\frac{(U^a)^{n+1} - (U^a)^n}{\Delta t} = \frac{1}{\sqrt{g^n}}\Big(\frac{1}{2}\sum_{l=1}^{2} A^n_{ll}(U^a)^{n+1} + \frac{1}{2}\sum_{l=1}^{2} A^n_{ll}(U^a)^n + \sum_{q=1}^{2}\sum_{r\neq q} A^n_{qr}(U^a)^n\Big),$$

where U^a is the N-dimensional vector denoting one of the components of the color vector, and A^n_{qr} is a central difference approximation of the operator $\partial_{x_q}(d_{qr}\partial_{x_r})$ at time step n.

Rearranging terms, we obtain

$$(U^a)^{n+1} = \left(I - \frac{\Delta t}{2\sqrt{g^n}}\sum_{l=1}^{2} A^n_{ll}\right)^{-1} \left(I + \frac{\Delta t}{\sqrt{g^n}}\sum_{q=1}^{2}\sum_{r\neq q} A^n_{qr} + \frac{\Delta t}{2\sqrt{g^n}}\sum_{l=1}^{2} A^n_{ll}\right)(U^a)^n,$$

which can also be written as

$$(U^a)^{n+1} = \left(I - \frac{\Delta t}{2}\sum_{l=1}^{2} \bar{A}^n_{ll}\right)^{-1} \left(I + \Delta t\sum_{q=1}^{2}\sum_{r\neq q} \bar{A}^n_{qr} + \frac{\Delta t}{2}\sum_{l=1}^{2} \bar{A}^n_{ll}\right)(U^a)^n,$$

where

$$\bar{A}_{11} = \frac{1}{\sqrt{g}}\partial_x(A\partial_x), \qquad \bar{A}_{22} = \frac{1}{\sqrt{g}}\partial_y(C\partial_y),$$

$$\bar{A}_{12} = \frac{1}{\sqrt{g}}\partial_x(B\partial_y), \qquad \bar{A}_{21} = \frac{1}{\sqrt{g}}\partial_y(B\partial_x),$$

and the functions A, B, C are the corresponding elements of the diffusion matrix associated with the Beltrami flow.

Again, this semi-implicit scheme still has a major drawback. At each iteration one needs to solve a large linear system whose matrix of coefficients is not tridiagonal and thus costly. Instead, we employ the LOD splitting scheme

$$(U^a)^{n+1} = \Big(I - \frac{\Delta t}{2}\bar{A}_{22}\Big)^{-1}\Big(I - \frac{\Delta t}{2}\bar{A}_{11}\Big)^{-1} \Big[(I + \frac{\Delta t}{2}\bar{A}_{11})(I + \frac{\Delta t}{2}\bar{A}_{22}) + \Delta t \sum_{q=1}^{2}\sum_{r\neq q}\bar{A}^n_{qr}\Big](U^a)^n,$$

or the AOS scheme, that reads,

$$(U^a)^{n+1} = \frac{1}{2}\Big[\,\big(I - \Delta t\bar{A}_{22}\big)^{-1} + \big(I - \Delta t\bar{A}_{11}\big)^{-1}\,\Big] \Big[(I + \frac{\Delta t}{2}\bar{A}_{11})(I + \frac{\Delta t}{2}\bar{A}_{22}) + \Delta t \sum_{q=1}^{2}\sum_{r\neq q}\bar{A}^n_{qr}\Big](U^a)^n.$$

The above splitting schemes are efficient because at each time step a single tridiagonal matrix inversion is performed for each spatial dimension.

The system of differential equations we deal with is nonlinear. The question of theoretical stability of the LOD/AOS based nonlinear finite difference scheme is a non-trivial challenge, with theory still lagging behind common practice. Our numerical experiments indicate that the splitting is stable for a wide variety of parameters, suitable for most applications, as will be shown in Section 5.

4.2 LOD/AOS scheme for the Beltrami-based denoising

The splitting scheme in the presence of a fidelity term requires a slight modification that we detail below. In this case we solve for each channel the equation

$$U^a_t = -\frac{2\lambda}{\sqrt{g}}(U^a - F^a) + \Delta_g U^a, \tag{7}$$

with von-Neumann boundary condition and the initial condition

$$U^a(x, 0) = F^a(x). \tag{8}$$

The Crank-Nicolson scheme approximating Eq. (7) is

$$(U^a)^{n+1} = \Big(I - \frac{\Delta t}{2}\sum_{l=1}^{2}\bar{A}^n_{ll} + 2\Delta t\frac{\lambda}{\sqrt{g^n}}I\Big)^{-1} \Big[\Big((I + \frac{\Delta t}{2}\bar{A}^n_{11})(I + \frac{\Delta t}{2}\bar{A}^n_{22}) + \Delta t \sum_{q=1}^{2}\sum_{r\neq q}\bar{A}^n_{qr}\Big)(U^a)^n + +2\Delta t F^a\frac{\lambda}{\sqrt{g^n}}\Big].$$

It is possible to use LOD/AOS approximations for the inverse of the matrix in the above scheme. However, we would like to treat the fidelity term in a special way. When $\lambda/\sqrt{g^n}$ is big, we find that the scheme proposed below possesses better stability properties.

We now describe the details for treating the fidelity term for our Crank-Nicolson scheme. Dividing the nominator and the denominator by the matrix $S^n = \left(1 + 2\Delta t \frac{\lambda}{\sqrt{g^n}}\right) I$, and rearranging terms, we get

$$
\begin{aligned}
(U^a)^{n+1} = & \Big(I - \frac{\Delta t}{2}(S^n)^{-1} \sum_{l=1}^{2} \bar{A}^n_{ll}\Big)^{-1} \\
& \Big[(S^n)^{-1}\Big((I + \frac{\Delta t}{2}\bar{A}^n_{11})(I + \frac{\Delta t}{2}\bar{A}^n_{22}) + \Delta t \sum_{q=1}^{2} \sum_{r \neq q} \bar{A}^n_{qr}\Big)(U^a)^n \\
& + 2(S^n)^{-1} \Delta t F^a \frac{\lambda}{\sqrt{g^n}}\Big].
\end{aligned}
$$

Approximating the semi-implicit scheme based on the LOD-splitting, we have

$$
\begin{aligned}
(U^a)^{n+1} = & \Big(I - \frac{1}{2}\Delta t (S^n)^{-1} \bar{A}^n_{22}\Big)^{-1} \Big(I - \frac{1}{2}\Delta t (S^n)^{-1} \bar{A}^n_{11}\Big)^{-1} \\
& \Big[(S^n)^{-1}\Big((I + \frac{\Delta t}{2}\bar{A}^n_{11})(I + \frac{\Delta t}{2}\bar{A}^n_{22}) + \Delta t \sum_{q=1}^{2} \sum_{r \neq q} \bar{A}^n_{qr}\Big)(U^a)^n + \\
& + 2(S^n)^{-1} \Delta t F^a \frac{\lambda}{\sqrt{g^n}}\Big].
\end{aligned}
$$

A similar splitting scheme can be developed using AOS.

5 Experimental results

We proceed to demonstrate experimentally the stability, accuracy, and efficiency of the LOD and AOS splitting schemes for the Beltrami color flow. In Figure 1 we show the results of the Beltrami flow, implemented by employing the LOD splitting scheme for approximating Eq. (4).

Next we illustrate the use of the splitting schemes in the case where the functional involves a fidelity term. A noisy image as well as the reference denoising result, based on the explicit scheme, are shown in Figure 3, next to the result of the AOS and LOD splitting schemes. Note that the visual results obtained by the two schemes are similar to the reference image.

5.1 RRE extrapolation technique for acceleration of the LOD splitting scheme

In [28, 1] vector extrapolation was applied in order to speed up the slow convergence of the explicit schemes for the Beltrami color flow. In the experiments

Fig. 1. Top row, left: The original image which contains JPEG artifacts. Middle: Results of the LOD splitting scheme with $\Delta t = 1$, after 1 iteration, $\beta = \sqrt{10^3}, \lambda = 0$. Right: Results of the LOD splitting scheme with after 2 iterations. Bottom row, left: Results of the LOD splitting scheme with after 4 iterations. Middle: a close-up of the original image. Right: a close-up of the resulting image after 4 iterations.

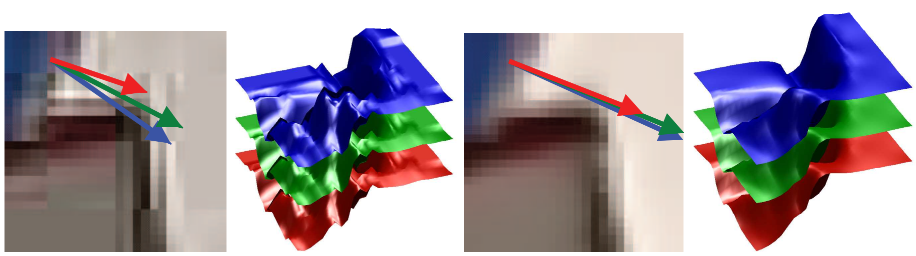

Fig. 2. The different image channels of an image patch taken from the images in Figure 1. Left to right: An image patch before denoising, its different color channels, the denoised image, and the denoised color channels. The color arrows indicate the direction of the gradient in the various color channels.

Fig. 3. Large image at the right: An image with artifacts resulting from lossy compression.. Smaller images – a close-up on a section of the image. Top row, left: The image with JPEG artifacts. Right: Beltrami-based denoising by explicit scheme, run with 4000 explicit iterations, $\Delta t = 0.0005$. Bottom row, left: Denoising by LOD, $\Delta t = 0.02$. Right: Denoising by AOS, $\Delta t = 0.02$. $\lambda = 1$, $\beta = \sqrt{2000}$.

below we demonstrate how the RRE extrapolation technique can also be used to accelerate the convergence of implicit schemes. Figure 4 shows that the RRE method accelerates the LOD scheme. A comparison is also given to the convergence rate achieved by the method of [28, 1]. Extrapolation techniques also allow us to obtain a more accurate rate, if one takes a smaller time step.

6 Conclusions

Due to its anisotropy and non-separability nature, no implicit scheme, nor operator splitting based scheme was so far introduced for the partial differential equations that describe the Beltrami color flow. In this paper we propose a semi-implicit splitting scheme based on LOD/AOS for the anisotropic Beltrami operator. The spatial mixed derivatives are discretized explicitly at time step $n\Delta t$, while the non-mixed derivatives are approximated using the average of the two time levels $n\Delta t$ and $(n + 1)\Delta t$.

The stability of the splitting is empirically tested in applications such as Beltrami-based scale-space and Beltrami-based denoising, which display a stable behavior. In order to further accelerate the convergence of the splitting schemes, the RRE vector extrapolation technique is employed.

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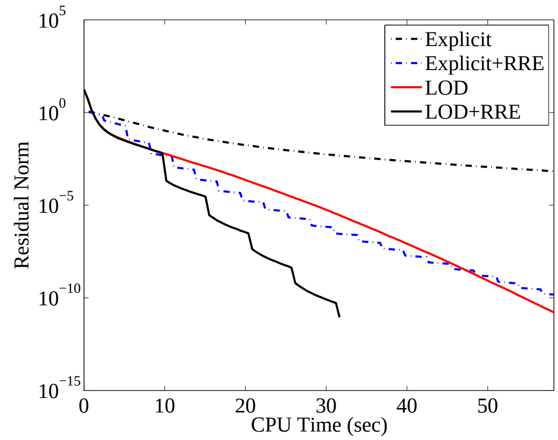

Fig. 4. Graph of the residuals (LOD, explicit+RRE and LOD+RRE) versus CPU times. Parameters: $\Delta t = 0.05$ for the explicit scheme, $\Delta t = 2.5$ for LOD, $\lambda = 0.5, \beta = \sqrt{500} \approx 22.36$.

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