The Robust Control of Robot Manipulators

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Abstract

A new scheme is presented for the accurate tracking control of robot manipulators. Based on the more general suction control methodology, the scheme addresses the following problem: Given the extent of parametric uncertainty (such as imprecisions or inertias, geometry, loads) and the frequency range of unmodeled dynamics (such as unmodeled structural modes, neglected time delays), design a nonlinear feedback controller to achieve optimal tracking performance, in a suitable sense. The methodology is compared with standard algorithms such as the computed torque method and is shown to combine in practice improved performance with simpler and more tractable controller designs.

1. Introduction

This paper presents a new scheme to achieve accurate tracking control of robot manipulators in the presence of model uncertainty and disturbances.

The development of efficient feedback control algorithms for robot manipulators has recently been the object of considerable interest (Brady et al. 1982). The complexity of the manipulator control problem largely reflects that of manipulator dynamics itself: Beyond three degrees of freedom, the derivation of manipulator dynamics becomes cumbersome to manage analytically and requires the use of sophisticated on-line computational algorithms (Luh, Waker, and Paul 1980*a*; Hollerbach 1980; Silver 1982; Renaud 1983). A major drawback of such algorithms is that they lack physical tractability, that is, they do not allow one to effectively exploit engineering insight during the design process (Bejczy and Lee 1983; Luh and Gu 1984). Further, the performance of standard control schemes based on these algorithms (for example, the computed torque or inverse method) is very sensitive to parametric uncertainty, that is, to imprecision on manipulator inertias, geometry, loads, and so on (Gilbert and Ha 1983).

In this paper, the *suction control* methodology (Slotine 1984) is proposed as a remedy to these drawbacks. For a class of nonlinear systems, suction control addresses the following problem: Given the extent of parametric uncertainty (such as imprecisions on inertias, geometry, loads) and of disturbances (such as Coulomb or viscous friction) and the frequency range of unmodeled dynamics (such as unmodeled structural modes, neglected time delays), design a nonlinear feedback controller to achieve optimal tracking precision, in a suitable sense. The explicit robustness guarantees provided by the methodology are shown to allow the use of simpler, more tractable manipulator models (such as "decoupled" arm and hand in a six-degree-of-freedom robot) while preserving stability in the face of model imprecision.

Section 2 summarizes the major features of suction controller design. The specific application of the methodology to the accurate tracking control of robot manipulators is described in Section 3. Section 4 concludes that the gains in engineering insight and controller simplicity offered by the methodology generally come for free because suction control schemes based on simplified models outperform standard algorithms based on higher-order models at most practical levels of parametric uncertainty.

2. Suction Control of Nonlinear Systems

In this section we review the basic results of Slotine and Sastry (1983) and Slotine (1984). For notational simplicity, the development is presented for systems with a single control input, although the extension to a large class of multi-input nonlinear systems is straightforward, as illustrated in an example. The specific design of suction controllers for robot manipulators is detailed in Section 3.

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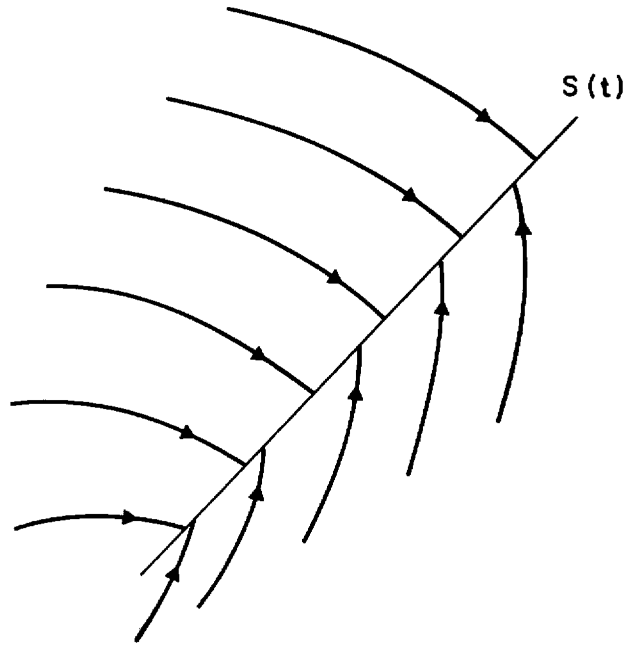
Fig. 1. The sliding condition.



2.1. Sliding Surfaces

Consider the dynamic system

$$x^{(n)}(t) = f(\mathbf{X};t) + b(\mathbf{X};t)u(t) + d(t), \qquad (1)$$

where $u(t)$ is the control input (say the applied torque at a manipulator joint) and $\mathbf{X} = [x, \dot{x}, \ldots, x^{(n-1)}]^T$ is the state. In Eq. (1), the function $f(\mathbf{X};t)$, in general nonlinear, is not exactly known, but the extent of the imprecision $|\Delta f|$ on $f(\mathbf{X};t)$ is upper-bounded by a known continuous function of $\mathbf{X}$ and t; similarly, control gain $b(\mathbf{X};t)$ is not exactly known but is of constant sign and is bounded by known, continuous functions of $\mathbf{X}$ and t. Disturbance $d(t)$ is unknown but upper-bounded by a known continuous function of $\mathbf{X}$ and t. The control problem is to get the state $\mathbf{X}$ to track a specific state $\mathbf{X}_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]^T$ in the presence of model imprecision on $f(\mathbf{X};t)$ and $b(\mathbf{X};t)$ and of disturbances $d(t)$. For this to be achievable using a finite control u, we must assume

$$\tilde{\mathbf{X}}|_{t=0} = 0, \qquad (2)$$

where

$$\tilde{\mathbf{X}} := \mathbf{X} - \mathbf{X}_d = [\tilde{x}, \dot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}]^T$$

is the tracking error vector.

We define a *time-varying sliding surface* $S(t)$ in the state-space $\mathbf{R}^n$ as

$$S(t) : s(\mathbf{X};t) = 0,$$

with

$$s(\mathbf{X};t) := \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}, \quad \lambda > 0, \qquad (3)$$

where λ is a positive constant (design parameter λ will later be interpreted as the desired control bandwidth). Given initial condition (2), the problem of tracking $\mathbf{X} \equiv \mathbf{X}_d$ is equivalent to that of remaining on the surface $S(t)$ for all $t > 0$. Now a sufficient condition for such positive invariance of $S(t)$ is to choose the control law u of Eq. (1) such that outside of sliding condition $S(t)$,

$$\frac{1}{2}\frac{d}{dt} s^2(\mathbf{X};t) \leq -\eta|s|, \qquad (4)$$

where η is a positive constant. Sliding condition (4) constrains state trajectories to point toward $S(t)$, as illustrated in Fig. 1.

The idea behind Eqs. (3) and (4) is to pick up a well-behaved function of the tracking error, s, according to Eq. (3) and then select the feedback control law u in Eq. (1) such that s^2 remains a Lyapunov function (Vidyasagar 1978) of the closed-loop system, in a suitable sense, despite the presence of model imprecision and disturbances. Further, satisfying Eq. (4) guarantees that if condition (2) is not exactly verified, that is, if $\mathbf{X}|_{t=0}$ is actually off $\mathbf{X}_d|_{t=0}$, the surface $S(t)$ will nonetheless be reached in a finite time (inferior to $s(\mathbf{X}(0); 0)/\eta$), while definition (3) then guarantees that $\tilde{\mathbf{X}} \rightarrow 0$ as $t \rightarrow \infty$.

The controller design procedure consists of two steps. First, a feedback control law u is selected so as to verify sliding condition (4). To account for the presence of modeling imprecision and of disturbances, however, such a control law is discontinuous across $S(t)$, which leads to control chattering. Chattering is undesirable in practice because it involves high control activity and further may excite high-frequency dynamics neglected in the course of modeling (such as unmodeled structural modes, neglected time delays, and the like). Thus, in a second step, discontinuous control law u is suitably smoothed to achieve an optimal trade-off between control bandwidth and tracking precision. While the first step accounts for parametric uncertainty, the second steps achieves robustness to high-frequency unmodeled dynamics.

Remark: The notion of sliding surface and the associated theory of variable structure systems (VSS) have been studied in great detail in the Soviet literature (see Utkin 1977 for a review). As discussed extensively in Slotine and Sastry (1983) and Slotine (1983), the VSS methodology has several important drawbacks, particularly high control authority and control chattering. An application of classical VSS theory to manipulator control is described in Young (1978).

2.2. Perfect Tracking Using Switched Control Laws

Given the bounds on uncertainties on $f(\mathbf{X};t)$ and $b(\mathbf{X};t)$ and on disturbances $d(t)$, constructing a control law to verify sliding condition (4) is straightforward, as we now illustrate on a simple multivariable example (numerous examples can be found in Slotine 1983).

Example: Consider the coupled multi-input system

$$\left[\begin{array}{ll} \ddot{\theta}_1 = a(t)(\theta_1)^3 + 2(\dot{\theta}_2)^2 + u_1 + d_1(t), & (5) \\ \ddot{\theta}_2 = f_2(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2; t) + u_2 + d_2(t), & (6) \end{array}\right.$$

where parameter $a(t)$ is estimated as $\hat{a}(t)$, with

$$|a(t) - \hat{a}(t)| \leq \alpha.$$

Consider the problem of getting θ_1 to track a desired trajectory θ_{d1}, specified in real time, such that

$$|\ddot{\theta}_{d1}(t)| \leq v, \quad \forall t \geq 0$$

in the presence of disturbances $d_1(t)$ such that

$$|d_1(t)| \leq \bar{d}, \quad \forall t \geq 0,$$

where v and $\bar{d}$ are known positive constants. We define the sliding surface $S_1(t)$ according to Eq. (3), namely

$$S_1(t) : s_1 = 0, \quad \text{with} \quad s_1 = \dot{\tilde{\theta}}_1 + \lambda\tilde{\theta}_1,$$

where $\tilde{\theta}_1 = \theta_1 - \theta_{d1}$ is the tracking error. Note that s_1 depends on θ_1 only. To satisfy a sliding condition of the form (4), we define control law u_1 as

$$u_1 = -\hat{a}(t)\theta_1^3 - 2\dot{\theta}_2^2 - \lambda\dot{\tilde{\theta}}_1 - (\alpha|\theta_1|^3 + \gamma)\ \mathrm{sgn}\ s_1, \qquad \gamma > \bar{d} + v, \tag{7}$$

where $\mathrm{sgn}(s) = -1$ for $s < 0$, and $\mathrm{sgn}(s) = +1$ for $s > 0$. Control law (7) is composed of a term $(-\hat{a}(t)\theta_1^3 - 2\dot{\theta}_2^2 - \lambda\dot{\tilde{\theta}}_1)$ that merely compensates for the known part of the dynamics of the variable s and of a term $(-[\alpha|\theta_1|^3 + \gamma]\ \mathrm{sgn}\ s_1)$, discontinuous across $S_1(t)$, that allows one to keep sliding condition (4) verified despite the presence of disturbances and parametric uncertainty. We have indeed

$$\frac{1}{2}\frac{d}{dt}(s_1)^2 = s_1 \cdot \dot{s}_1 = s_1 \cdot (\ddot{\theta}_1 - \ddot{\theta}_{d1} + \lambda\dot{\tilde{\theta}}_1),$$

so control law (7) applied to system dynamics (5) leads to

$$\frac{1}{2}\frac{d}{dt}(s_1)^2 = s_1 \cdot [(a(t) - \hat{a}(t))\theta_1^3 - \ddot{\theta}_{d1}(t) + d_1(t)] - |s_1| \cdot [\alpha|\theta_1|^3 + \gamma],$$

where

$$\alpha \geq |a(t) - \hat{a}(t)|, \quad \gamma > v + \bar{d} \geq |-\ddot{\theta}_{d1} + d_1(t)|.$$

Hence this satisfies sliding condition (4) for surface $S_1(t)$:

$$\frac{1}{2}\frac{d}{dt}(s_1)^2 \leq -[\gamma - (v + \bar{d})] \cdot |s_1|.$$

Note that control discontinuity at $S_1(t)$ increases with parametric imprecision and strength of disturbances to be compensated for. Further, if desired acceleration $\ddot{\theta}_{d1}(t)$ is explicitly available, we may rather use

$$u_1 = \ddot{\theta}_{d1}(t) - \hat{a}(t)\theta_1^3 - 2\dot{\theta}_2^2 - \lambda\dot{\tilde{\theta}}_1 - (\alpha|\theta_1|^3 + \gamma)\mathrm{sgn}\ s_1, \quad \gamma > \bar{d}$$

instead of Eq. (7), thus reducing control discontinuity at $S_1(t)$.

Assume now that Eq. (5) is replaced by

$$\ddot{\theta}_1 = a(t)\theta_1^3 + 2\dot{\theta}_2^2 + b_1(t)u_1 + d_1(t), \tag{8}$$

where gain $b_1(t)$ is estimated by $\hat{b}_1(t)$ such that

$$\frac{1}{\beta_1} \leq \frac{\hat{b}_1(t)}{b_1(t)} \leq \beta_1, \quad \beta_1 \geq 1. \tag{9}$$

Parameter β_1 in Eq. (9) can be interpreted as the *gain margin* on control law u_1 relative to (possibly time-varying or state-dependent) control gain estimate $\hat{b}_1(t)$. Letting

$$u_{10} := \hat{a}(t)\theta_1^3 + 2\dot{\theta}_2^2 + \lambda\dot{\tilde{\theta}}_1,$$

we can satisfy condition (4) by now choosing

$$u_1 = -\frac{1}{\hat{b}_1(t)}\left[\frac{2u_{10}}{\beta_1 + (1/\beta_1)} + \beta_1\left(\frac{|u_{10}|[\beta_1 - (1/\beta_1)]}{\beta_1 + (1/\beta_1)} + \alpha|\theta_1|^3 + \gamma\right)\operatorname{sgn} s_1\right], \quad \gamma > \bar{d} + v \tag{10}$$

instead of (7). We thus have further increased control discontinuity at $S_1(t)$ to compensate for uncertainty on gain $b_1(t)$ (note that the choice of u_1 to satisfy sliding condition 4 is not unique; expression 10 minimizes the continuous part of $\dot{s}_1$).

Notice the decoupling in the design: If θ_2 were also prescribed to track a given trajectory, one would define $s_2 := \dot{\tilde{\theta}}_2 + \lambda_2\tilde{\theta}_2$ and repeat the procedure independently.

As illustrated in the preceding example, and shown in further detail in Slotine and Sastry (1983), the methodology can easily be extended to multivariable systems of the form

$$x_j^{(n_j)} = f_j(\mathbf{X}_1, \ldots, \mathbf{X}_m;t) + b_j(\mathbf{X}_1, \ldots, \mathbf{X}_m;t)u_j + d_j(t), \quad j = 1, \ldots, m, \tag{11}$$

where $\mathbf{X}_j = [x_j, \dot{x}_j, \ldots, x_j^{(n_j-1)}]^T$. We assume again that disturbances d_j and imprecisions on f_j and b_j are bounded by known continuous functions of $\mathbf{X}_j$ and of time. Dynamics of the form (11) describe a large number of nonlinear systems encountered in practice, including a vast class of mechanical systems (from Lagrange's equations). Further, Hunt, Su, and Meyer (1983) have shown that a wide class of controllable nonlinear systems could be put in the form (11) by using appropriate "global" transformations: Being able to deal explicitly with imprecision on the system model allows for numerical conditioning problems that may affect such transformations to be easily accounted for.

2.3. Suction Control

2.3.1. Continuous Control Laws to Approximate Switched Control

As seen in the example of Section 2.2, control laws that satisfy sliding condition (4) are discontinuous across the surface $S(t)$, thus leading to control chattering. Chattering is in general highly undesirable in practice because it involves extremely high control activity and further may excite high-frequency dynamics neglected in the course of modeling. We can remedy this situation by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface (see Fig. 2):

$$B(t) = \{\mathbf{X}, |s(\mathbf{X};t)| \leq \Phi\}, \quad \Phi > 0, \tag{12}$$

where Φ is the boundary layer thickness and $\epsilon := \Phi/\lambda^{n-1}$ is the boundary layer width. This is achieved by choosing outside of $B(t)$ control law u as before—that is, satisfying sliding condition (4), which guarantees boundary layer attractiveness, and hence positive invariance; all trajectories starting inside $B(t = 0)$ remain inside $B(t)$ for all $t \geq 0$—and then interpolating u inside $B(t)$—for instance, replacing in the expression of u the term sgn s by s/Φ, inside $B(t)$, as illustrated in Figs. 2 and 3. As proved in Slotine (1983), this leads to tracking to within a guaranteed precision ϵ and more generally guarantees that for all trajectories starting inside $B(t = 0)$,

$$|\tilde{x}^{(i)}(t)| \leq (2\lambda)^i\epsilon, \quad i = 0, \ldots, n-1.$$

These bounds are understood asymptotically, with a time constant $(n-1)/\lambda$; they hold for all $t \geq 0$ if $\tilde{\mathbf{X}}|_{t=0} = 0$. We now show that the smoothing of control

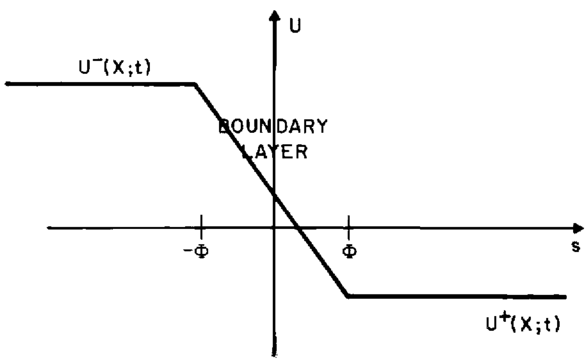


Fig. 2. Construction of the boundary layer in the case that n = 2.

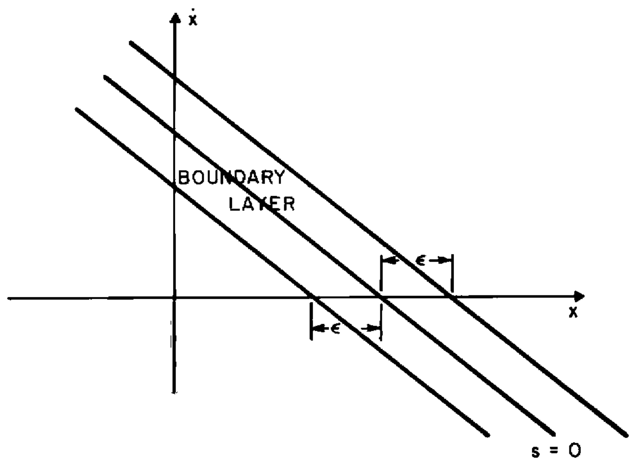


Fig. 3. Sample interpolation of the control law in the boundary layer.

discontinuity inside $B(t)$ essentially assigns a lowpass filter structure to the local dynamics of the variable s, thus eliminating chattering. Recognizing this filter structure allows us to tune up the control law so as to achieve a trade-off between tracking precision and robustness to unmodeled dynamics.

For clarity, we first consider the case of no gain margin ($\beta = 1$) and then generalize. Further details on the following development can be found in Slotine (1984).

Consider the system

$$X^{(n)} = f(\mathbf{X};t) + u + d(t), \tag{13}$$

where

$$f(\mathbf{X};t) = \hat{f}(\mathbf{X};t) + \Delta f(\mathbf{X};t). \tag{14}$$

In Eq. (14), $\hat{f}(\mathbf{X};t)$ is the available model of $f(\mathbf{X};t)$. Further, as in Section 2, we assume

$$|\Delta f(\mathbf{X};t)| \leq F(\mathbf{X};t), \tag{15}$$

$$|d(t)| \leq D(\mathbf{X};t), \tag{16}$$

where F, D, and $\hat{f}$ are known continuous functions of $\mathbf{X}$ and t; uncertainty $\Delta f(\mathbf{X};t)$ on dynamics $f(\mathbf{X};t)$ is assumed to be continuous in $\mathbf{X}$.

The control problem is to get the state $\mathbf{X}$ to track a desired state $\mathbf{X}_d$, specified in real time, such that a priori:

$$|x_d^{(n)}(t)| \leq v(t),$$

although $x_d^{(n)}(t)$ itself is not explicitly available. Defining $s(\mathbf{X};t)$ according to Eq. (3) and letting

$$k(\mathbf{X};t) := F(\mathbf{X};t) + D(\mathbf{X};t) + v(t) + \eta \geq \eta > 0, \tag{17}$$

where η is a positive constant, the control law

$$u = -\hat{f}(\mathbf{X};t) - \sum_{p=1}^{n-1} \binom{n-1}{p} \lambda^p \tilde{x}^{(n-p)} - k(\mathbf{X};t)\,\mathrm{sgn}(s) \tag{18}$$

satisfies sliding condition (4).[1] As in the example of Section 2.2, control law (18) is composed of a term,

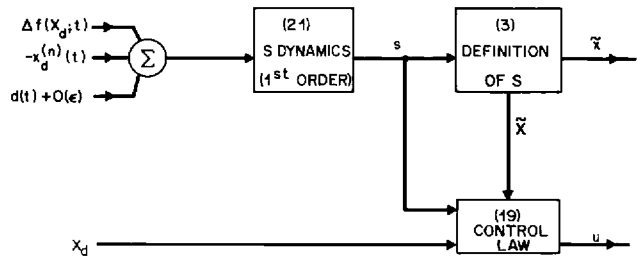
$$-\hat{f}(\mathbf{X};t) - \sum_{p=1}^{n-1} \binom{n-1}{p} \lambda^p \tilde{x}^{(n-p)},$$

that compensates for the known part of the dynamics of the variable s and of a term $-k(\mathbf{X};t)\,\mathrm{sgn}(s)$, discontinuous across $S(t)$, that keeps sliding condition (4) verified in spite of parametric uncertainty and disturbances. Let us now smooth out the control discontinuity inside the boundary layer $B(t)$ of thickness Φ, defined by Eq. (12). Control law (18) then becomes

$$u = -\hat{f}(\mathbf{X};t) - \sum_{p=1}^{n-1} \binom{n-1}{p} \lambda^p \tilde{x}^{(n-p)} - k(\mathbf{X};t)\,\mathrm{sat}(s/\Phi), \tag{19}$$

1. $\begin{bmatrix} n \\ m \end{bmatrix} = \dfrac{n!}{m!(n-m)!}$ for $m \leq n$.

Fig. 4. Dynamic structure of the closed-loop system.



where the saturation function sat is defined by

$$|y| \leqslant 1 \Rightarrow \text{sat}(y) = y; |y| > 1 \Rightarrow \text{sat}(y) = \text{sgn}(y).$$

Since by construction control u satisfies Eq. (4) outside of $B(t)$, the boundary layer is attractive and hence (positively) invariant. Thus, for trajectories starting inside $B(t = 0)$ (in particular, if $\tilde{\mathbf{X}}|_{t=0} = 0$) we can write from Eqs. (3) and (19), for all $t \geqslant 0$,

$$\dot{s} = -k(\mathbf{X};t)[s/\Phi] + (\Delta f(\mathbf{X};t) + d(t) - x_d^{(n)}(t)). \quad (20)$$

Moreover, since by construction a tracking error of ϵ is achieved, we can rewrite Eq. (20) as

$$\dot{s} = -k(\mathbf{X}_d;t)[s/\Phi] + (\Delta f(\mathbf{X}_d;t) + d(t) - x_d^{(n)}(t) + 0(\epsilon)), \quad (21)$$

since $\Delta f(\mathbf{X};t)$ and $k(\mathbf{X};t)$ are continuous in $\mathbf{X}$. We see from Eq. (21) that the variable s, which is a measure of the algebraic distance to the surface $S(t)$, is the output of a stable first-order filter whose dynamics depend only on the desired state $\mathbf{X}_d(t)$ and perhaps explicitly on time and whose inputs are (to first order) perturbations: disturbance $d(t)$, uncertainty $\Delta f(\mathbf{X}_d;t)$ on the dynamics, and nth derivative $x_d^{(n)}$ of the desired trajectory.

Equation (21) shows that chattering is indeed eliminated, as long as high-frequency unmodeled dynamics are not excited. The dynamic structure of the closed-loop system is summarized in Fig. 4: perturbations are filtered according to Eq. (21) to give s, which in turn provides tracking error $\tilde{x}$ by further lowpass filtering, according to definition (3); control u is a function of s, $\mathbf{X}$ and $\mathbf{X}_d$ as specified by Eq. (19). Since λ is the break frequency of the filter described by Eq. (3), it must be chosen to be "small" with respect to high-frequency unmodeled dynamics (such as unmodeled structural modes or neglected time delays). Assume now that $F(\mathbf{X}_d;t)$, $D(\mathbf{X}_d;t)$, and $v(t)$ can be a priori upper-bounded so that $k(\mathbf{X}_d;t)$ can be upper-bounded, say by k_{max}. The constant k_{max}/Φ may be thought of as the "break frequency" of Eq. (21); as λ, it must also be chosen to be small with respect to high-frequency unmodeled dynamics. Thus, if λ is set to be the largest acceptable break frequency of Eq. (3), we must have

$$k_{max}/\Phi \leqslant \lambda,$$

which fixes the best attainable tracking precision ϵ:

$$\lambda\Phi = k_{max}, \quad \text{that is,} \quad \lambda^n\epsilon = k_{max}. \quad (22)$$

We will refer to Eq. (22) as the *balance condition* for the control system. Intuitively, it amounts to tuning up the control law so that the closed-loop system will be as close as possible to "critical damping" (if $k(\mathbf{X}_d;t)$ were constant, Eq. 22 would exactly correspond to critical damping). Of course, desired trajectory $x_d(t)$ itself must be slow with respect to unmodeled dynamics.

The preceding development was for $\beta = 1$ (no gain margin). Assume now that Eq. (13) is replaced by

$$x^{(n)} = f(\mathbf{X};t) + b(\mathbf{X};t)u + d(t),$$

where gain $b(\mathbf{X};t)$ is estimated by $\hat{b}(\mathbf{X};t)$, with a gain margin β:

$$1/\beta \leqslant \hat{b}(\mathbf{X};t)/b(\mathbf{X};t) \leqslant \beta.$$

If, instead, $\beta_{min} \leqslant \hat{b}(\mathbf{X};t)/b(\mathbf{X};t) \leqslant \beta_{max}$, set $\beta := \sqrt{\beta_{max}/\beta_{min}}$ and use $\hat{b}_{new}(\mathbf{X};t) := \hat{b}(\mathbf{X};t)/\sqrt{\beta_{min}\beta_{max}}$ as the new estimate of $b(\mathbf{X};t)$; note that β_{min}, β_{max}, and β can be time-dependent. One can then show that the balance condition is now of the form

$$\lambda\Phi = \beta k_{max}, \quad \text{that is,} \quad \lambda^n\epsilon = \beta k_{max}, \quad (23)$$

instead of Eq. (22). Thus, having chosen λ to be small with respect to high-frequency unmodeled dynamics, the balance condition (23) fixes the best attainable tracking precision ϵ, given bandwidth requirements and a choice of $k(\mathbf{X};t)$ accounting for modeling uncertainties, disturbances to be rejected, and range of trajectories to be tracked. Further, as shown in Slotine (1983), extension of the balance condition to the class of multivariable systems in Eq. (10) is straightforward: for each degree of freedom $j (j = 1, \ldots, m)$ we obtain a condition of the form

$$\lambda_j \Phi_j = \beta_j k_{j\max}, \quad \text{that is} \quad (\lambda_j)^{n_j} \epsilon_j = \beta_j k_{j\max}, \quad (24)$$

where in general $k_{j\max}$ depends on all $\mathbf{X}_{d1}, \ldots, \mathbf{X}_{dm}$.

Example: Consider again the system (8) of Section 2.2, with (discontinuous) control law u_1 defined as in Eq. (10). The corresponding gain $k_1(\mathbf{X};t)$ is

$$k_1(\mathbf{X};t) = \beta_1(\zeta|u_{10}| + \alpha|\theta_1|^3 + \gamma),$$

where $\zeta := (\beta_1 - 1/\beta_1)/(\beta_1 + 1/\beta_1)$, so that

$$k_1(\mathbf{X}_d;t) = \beta_1(\zeta|\hat{a}(t)\theta_{d1}^3 + 2\dot{\theta}_{d2}^2| + \alpha|\theta_{d1}|^3 + \gamma).$$

Assume now that

$$|\theta_{d1}| \leq \theta_{d1}^{\max}, \quad |\dot{\theta}_{d2}| \leq \dot{\theta}_{d2}^{\max}, \quad |\hat{a}(t)| \leq a^{\max}, \quad \forall t \geq 0.$$

Then

$$k_1(\mathbf{X}_d;t) \leq k_1^{\max} := \beta_1[\zeta \cdot (a^{\max}(\theta_{d1}^{\max})^3 + 2(\dot{\theta}_{d2}^{\max})^2) + \alpha|\theta_{d1}^{\max}|^3 + \gamma].$$

Assume further that the system exhibits an unmodeled structural mode at $\nu_{\text{structure}} \approx 5$ Hz. A reasonable choice for λ may then be, for instance,

$$\lambda := 10 \text{ rad/s} < (2\pi/3) \cdot \nu_{\text{structure}},$$

which then fixes tracking precision ϵ_1:

$$\epsilon_1 := \beta_1 k_{1\max}/\lambda^2 = \beta_1 k_{1\max}/100.$$

Continuous control law u_1 is then obtained by replacing the term sgn s_1 by sat$(s_1/\lambda\epsilon_1)$ in expression (10).

Remarks:

1. Effect of data sampling can also be interpreted as part of high-frequency unmodeled dynamics. It can be shown that the corresponding "soft" upper bound that sampling rate ν_{sampling} imposes on λ is

$$2(1 + \tau_{\text{process}} \cdot \nu_{\text{sampling}})\lambda < \nu_{\text{sampling}},$$

where $(\tau_{\text{process}} \cdot \nu_{\text{sampling}})$ is the ratio of processing delay over sampling period. For instance, in the case of a full-period processing delay we must choose

$$\lambda < \nu_{\text{sampling}}/4.$$

2. By an argument similar to that of the above discussion, it can be shown that the specific choice of the dynamics of Eq. (3) used to define sliding surfaces is the best conditioned among linear dynamics, in the sense that it guarantees the best tracking precision ϵ given the desired control bandwidth and the extent of parametric uncertainty and disturbances. Conversely, the definition of the sliding surface can be shaped to address more general applications than tracking control. For instance, in the context of compliance control (Hogan 1984), $s = 0$ can be chosen to describe a desired dynamic response to measured forces exerted by the environment. The methodology developed in this paper largely extends to such problems and in particular represents an effective and systematic procedure to reduce sensitivity to modal uncertainty in compliance control schemes for robot manipulators.

2.3.2. *The Dynamic Balance Conditions*

The above trade-off between robustness and tracking precision, quantified by balance conditions (24), can be further improved by allowing boundary layer widths to be time-dependent.

In Section 2.3.1, by seeking the smallest constant ϵ such that

$$\beta k(\mathbf{X}_d;t)/(\lambda^{n-1}\epsilon) \leq \lambda, \quad \forall k(\mathbf{X}_d;t),$$

we obtained the static balance condition (22):

$$\lambda^n \epsilon = \beta k_{\max}.$$

If we now allow boundary layer width ϵ to be time-varying, we expect to be able to refine the balance condition, that is, to define a boundary layer width history $\epsilon(t)$ such that

$$\epsilon(t) \leqslant \beta k_{\max}/\lambda^n, \quad \forall t \geqslant 0, \tag{25}$$

while preserving system bandwidth λ However, to maintain boundary attractiveness while allowing for variations of boundary layer thickness $\Phi = \lambda^{n-1}\epsilon$, we must now choose the control law u such that outside of $B(t)$

$$\frac{1}{2}\frac{d}{dt}s^2 \leqslant (\dot{\Phi} - \eta)|s|, \quad \eta > 0, \tag{26}$$

instead of Eq. (4). The additional term $\dot{\Phi}|s|$ in Eq. (26) reflects the fact that the boundary layer attractiveness condition is more stringent during boundary layer contraction ($\dot{\Phi} < 0$) and less stringent during boundary layer expansion ($\dot{\Phi} > 0$).

It is shown in Slotine (1984) that, to satisfy condition (26) while preserving control bandwidth λ, one can simply replace gain $k(\mathbf{X};t)$ of Eq. (19) by a new gain $\bar{k}(\mathbf{X};t)$, where

$$\bar{k}(\mathbf{X};t) := [k(\mathbf{X};t) - k(\mathbf{X}_d;t)] + \lambda\Phi/\beta, \tag{27}$$

with static balance condition (23) replaced by the *dynamic balance conditions*

$$\left[\begin{array}{ll} k(\mathbf{X}_d;t) \geqslant \lambda\Phi/\beta \Rightarrow \dot{\Phi} + \lambda\Phi = \beta k(\mathbf{X}_d;t), & (28) \\ k(\mathbf{X}_d;t) \leqslant \lambda\Phi/\beta \Rightarrow \dot{\Phi} + \lambda\Phi/\beta^2 = k(\mathbf{X}_d;t)/\beta, & (29) \end{array}\right.$$

with, initially,

$$\Phi(0) := \beta k(\mathbf{X}_d(0); 0)/\lambda. \tag{30}$$

This particular type of sliding control, which uses time-varying boundary layer widths to account for time dependence of parametric uncertainty, will be referred to as *suction control.* It is often the case in practice that $k(\mathbf{X}_d;t)$ shows large variations along a desired trajectory. In these instances, the use of time-varying boundary layer widths, as specified by the dynamic balance conditions, greatly improve tracking performance while introducing only modest additional complexity. In robot manipulator control, for instance, $k(\mathbf{X};t)$ involves centripetal terms $\dot{\theta}_i^2$ and Coriolis terms $\dot{\theta}_i\dot{\theta}_j$ (where the θ_i are joint angles). The dynamic balance conditions then allow one to efficiently trade off speed against tracking precision while preserving system robustness to unmodeled dynamics. Also, recalling that $k(\mathbf{X};t)$ reflects uncertainty on system dynamics, $k(\mathbf{X}_d;t)$ may be decreased as the result of a parameter estimation process, for instance. The dynamic balance conditions allow one to easily account for such on-line improvement on modeling precision. The suction control methodology is thus likely to provide robust "adaptive" schemes because it guarantees stability and fixes control system bandwidth while achieving best tracking precision given current modeling uncertainties.

Further, in some practical instances, desired bandwidth itself may vary with time. In robot manipulator control, for example, structural resonant frequencies decrease as the load mass at the tip of the arm gets larger (Paul 1981). The control law, initially tuned not to excite the lowest expectable mode (that is, to handle maximum load), can thus exploit on-line load estimation by increasing control bandwidth in addition to decreasing $k(\mathbf{X}_d;t)$. Further, structural resonant frequencies actually vary with manipulator configuration (although because of its complexity this effect is rarely modeled). Similarly, it is desirable to monitor mechanical compliance when performing automatic assembly of close-tolerance parts (Hogan 1981; Asakawa, Akiya, and Tabata 1982). Let us call λ_0 the constant value of λ that we previously used, that is, the desired control bandwidth based on a uniform lower bound on frequencies of unmodeled dynamics. It can be shown that Eqs. (27)–(29) remain valid for time-varying $\lambda = \lambda(t)$, provided control u is now defined as

$$u := -\hat{f}(\mathbf{X};t) - \sum_{p=1}^{n-1} \binom{n-1}{p} \lambda^{p-1}(\lambda\tilde{x}^{(n-p)} + p\dot{\lambda}\tilde{x}^{(n-p-1)}) - \bar{k}(\mathbf{X};t)\,\mathrm{sat}(s/\Phi), \tag{31}$$

instead of Eq. (19). Time derivative $\dot{\lambda}$ in Eq. (31) must be chosen smooth enough not to excite high-frequency unmodeled dynamics. A practical way to do so is to generate $\lambda(t)$ by filtering desired bandwidth value λ_{desired} through a second-order critically damped filter of break frequency λ_0. Of course, initial condition (30) is now replaced by

$$\Phi(0) := \beta k(\mathbf{X}_d(0){:}0)/\lambda(0). \tag{32}$$

Remarks:

1. Since ϵ essentially varies as λ^{-n} for given parametric uncertainty, bandwidth variations are strongly reflected in tracking performance.
2. In the case that $\beta = 1$ (no gain margin), Eqs. (28)–(29) reduce to

$$\dot{\Phi} + \lambda\Phi = k(\mathbf{X}_d;t).$$

3. When the dynamics of $k(\mathbf{X}_d;t)$ is slow with respect to frequency λ_0/β^2, one can simply use

$$\bar{k}(\mathbf{X};t) := k(\mathbf{X};t),$$
$$\Phi := \beta k(\mathbf{X}_d;t)/\lambda(t),$$

   instead of Eqs. (27)–(29).
4. The preceding development remains valid in the case that gain margin β is time-varying.
5. The s-trajectory, that is, the variation of s/Φ with time, is a compact descriptor of the closed-loop system behavior: Control activity directly depends on s/Φ, while by definition (3) tracking error $\tilde{x}$ is merely a filtered version of s. Further, the s-trajectory represents a time-varying measure of the validity of the assumptions on model uncertainty. Similarly, boundary layer thickness Φ describes the evolution of dynamic model uncertainty with time. The use of s-trajectories for multivariable control system design and testing is demonstrated in Slotine, Yoerger, and Sheridan, in press).

2.4. Summary

By controlling the system along time-varying sliding surfaces in the state-space, we achieved perfect tracking of desired trajectories for a class of multivariable nonlinear time-varying systems.

By substituting smooth transitions across a boundary layer to control switching at the sliding surface, we eliminated chattering and obtained a trade-off between tracking precision and robustness to unmodeled dynamics.

By allowing boundary layer width to be time-varying, we refined the above trade-off to account for time-dependence of parameter uncertainties, thus improving tracking precision while still maintaining robustness to high-frequency unmodeled dynamics.

Finally, we monitored the orientation of the boundary layer in the state-space (defined by λ) to account for possible time dependence of desired bandwidth (whether due to actual changes in the plant or to online modeling improvements), thus further improving tracking performance.

We now describe more specifically the application of the methodology to the feedback control of robot manipulators. The scheme is compared to standard algorithms such as the computed torque method and is shown to significantly simplify controller design, thus permitting us to retain and exploit engineering insight all along the design and implementation process.

3. The Robust Control of Robot Manipulators

3.1. Robustness Issues in Manipulator Control

There has recently been a considerable interest in developing efficient control algorithms for robot manipulators (Brady et al. 1982). The complexity of the control problem for manipulators arises mainly from that of manipulator dynamics itself: The dynamics of articulated mechanisms in general, and of robot manipulators in particular, involves strong coupling effects between joints as well as centripetal and Coriolis forces (particularly significant at high speeds). The equations of motion for an n-link manipulator may be expressed as follows (Paul 1981):

$$\mathbf{R}(\theta)\ddot{\theta} + \mathbf{h} = \mathbf{T}, \tag{33}$$

where θ is the n-vector of joint angles (or more generally of joint displacements), T is the vector of applied torques (or generalized forces), $\mathbf{R}(\theta)$ is the inertia matrix (symmetric, positive definite) that reflects coupling effects between joints, and $\mathbf{h} = \mathbf{h}(\theta, \dot{\theta}; t)$ contains centripetal and Coriolis forces as well as friction and gravitational terms. For an articulated manipulator, the dynamics is reasonably manageable with pencil and paper only up to three degrees of freedom (Horn, Hirokawa, and Varizani 1977). Thus, much effort has been devoted to developing efficient procedures for real-time computation of the dynamics. A substantial improvement in computational efficiency was obtained by using recursive algorithms (Hollerbach 1980; Luh, Walker, and Paul 1980*a*; Silver 1982; Renaud 1983) to generate the torques required to support a desired motion (inverse dynamics), achieving linear variation of the computation complexity with the number of links.

While the inverse dynamics computes the torques theoretically needed to compensate for nonlinearities and to follow a specific trajectory, assuming an exact model, explicit values of inertia matrix $\mathbf{R}(\theta)$ and nonlinear terms $\mathbf{h}$ are required to analyze how joint accelerations (hence subsequent motion) are affected not only by control torques and known dynamics but also by disturbances (such as Coulomb and viscous friction) and modeling errors: parametric uncertainties such as inaccuracies on inertias, geometry, loads; and high-frequency unmodeled dynamics, such as unmodeled structural modes or neglected time delays. Currently, the most efficient algorithm to compute $\mathbf{R}(\theta)$ and $\mathbf{h}$ (in fact, to estimate these terms, precisely because of parametric uncertainties) is due to Walker and Orin (1982) and uses the inverse recursive dynamics formulation of Luh, Walker, and Paul (1980*a*) as an important component. The algorithm could be further simplified by exploiting geometric features of "well-structured" manipulators, using the results of Hollerbach and Sahar (1983), and even further by customizing the computations for a specific robot, along the lines of Kanade, Khosla, and Tanaka (1984).

Now there are several reasons for insisting on explicit robustness guarantees for the robot control system. The first is obvious: Control instabilities are unpleasant, especially at high speeds (an elegant stability analysis of computed-torque-like schemes can be found in Gilbert and Ha 1983). Conversely, guaranteed robustness properties allow one to design simple controllers. Consider, for instance, a six-degree-of-freedom manipulator composed of a three-degree-of-freedom arm and a three-degree-of-freedom hand. Kinematic decoupling between hand and arm can be achieved by having the three rotational axe of the wrist intersect at a point (spherical wrist). Physically it seems natural to seek a similar decoupling at the dynamic level, in other words to be able to consider motions of arm and hand as "disturbances" to one another (each of these disturbances being possibly further decomposed into an average, quickly estimated term to be directly compensated for and a genuine perturbation term to be accounted for by the controller robustness). Further, desired bandwidth is likely to be much larger for the hand than for the arm itself (Salisbury 1984). Robust controller design does allow such natural reduction of the original problem into two lower-order control problems: In the case of a six-degree-of-freedom manipulator, both of these problems are amenable to closed-form, pencil-and-paper treatment, thus allowing one to maintain clear physical insight and exploit engineering judgment all along the design and implementation process.[2] Finally, by easily accounting for large model imprecision, robust controllers simplify higher-level programming.

The relevance of suction controllers to robot manipulator control is precisely that they feature such explicit, built-in robustness properties. Depending on the structure of model uncertainty and on the type of user space considered (configuration space, task-oriented or hybrid (force/position) coordinates, and so on), the suction control ideas of Section 2 may be applied in various ways. The following scheme assumes for simplicity that desired trajectories are specified in joint space. Also, as discussed above, if N is the total number of degrees of freedom of the manipulator and N_H is the number of degrees of freedom of the wrist and hand, the first $N - N_H$ and the last N_H degrees of freedom of the arm can be treated separately (for a wrist-partitioned robot). Thus, in the sequel, system size n refers to either $N - N_H$ or N_H.

2. The importance of model tractability has been stressed by several authors in a different context; see, for example, Bejczy and Lee (1983) and Luh and Gu (1984).

3.2. Suction Controller Structure

Let λ be the desired control bandwidth, chosen small enough not to excite unmodeled structural modes or interefere with neglected time delays (see the example of Section 2.3.1), and let

$$\hat{\mathbf{R}}(\theta)\ddot{\theta} + \hat{\mathbf{h}} = \mathbf{T} \tag{34}$$

be the available model of the manipulator described by Eq. (33). Discrepancies between Eqs. (33) and (34) may arise from several factors: imprecisions on the manipulator geometry or inertias, uncertainties on the friction terms or the loads, on-line computation limitations, or purposeful model simplification, as discussed earlier. The suction controller for the manipulator of Eq. (33) takes the form

$$\mathbf{T} = \hat{\mathbf{R}}\mathbf{u} + \hat{\mathbf{h}}, \tag{35}$$

where the components u_i ($i = 1, \ldots, n$) of the vector **u** are defined as

$$u_i := \ddot{\theta}_{di} - 2\lambda\dot{\tilde{\theta}}_i - \lambda^2\tilde{\theta}_i - G_i(\theta)\bar{k}_i(\theta, \dot{\theta}; t)\,\mathrm{sat}(s_i/\Phi_i). \tag{36}$$

In Eq. (36), $\ddot{\theta}_{di}$ is the acceleration of the desired trajectory at joint i, $\tilde{\theta}_i$ is the tracking error at joint i, and $\bar{k}_i(\theta, \dot{\theta}; t)$ and Φ_i are defined according to the dynamic balance conditions (27)–(30) of Section 2 (the scaling factor $G_i(\theta)$ of Eq. (36) and the bounds on model uncertainty used in the computation of $\bar{k}_i(\theta, \dot{\theta}; t)$ will be discussed next). The surfaces s_i in Eq. (36) are set to be

$$s_i := \dot{\tilde{\theta}}_i + 2\lambda\tilde{\theta}_i + \lambda^2 \int_0^t \tilde{\theta}_i(\tau)d\tau = \left(\frac{d}{dt} + \lambda\right)^2 \left(\int_0^t \tilde{\theta}_i(\tau)d\tau\right) \tag{37}$$

so as to use integral control and reduce the effects of friction. It is important to remark that the major difference with the computed-torque method is the presence of the robustifying terms in $G_i(\theta)\,\bar{k}_i(\theta, \dot{\theta}; t)$ $\mathrm{sat}(s_i/\Phi_i)$ in Eq. (36), which allow one to maintain stability and optimize performance in the face of model uncertainty.

3.3. Practical Evaluation of Parametric Disturbances

There remains only to evaluate explicit bounds on parametric uncertainty so as to generate a set of control discontinuity gains $k_i(\theta, \dot{\theta}; t)$, as in Section 2; these gains will then be fed into the dynamic balance conditions (27)–(30) to compute modified gains $\bar{k}_i(\theta, \dot{\theta}; t)$ and boundary layer thicknesses Φ_i of Eq. (36). A simplified, easily implementable approach to such evaluation is as follows: Define two sets of n-vectors, $\{\mathbf{L}_j; j = 1, \ldots, n\}$ and $\{\Delta\mathbf{R}_j; j = 1, \ldots, n\}$ by

$$\mathbf{R}^{-1} =: [\mathbf{L}_1 \cdots \mathbf{L}_n]$$
$$\hat{\mathbf{R}} - \mathbf{R} =: \Delta\mathbf{R} =: [\Delta\mathbf{R}_1 \cdots \Delta\mathbf{R}_n]$$

The vectors $\mathbf{L}_j$ and $\Delta\mathbf{R}_j$ are functions of configuration θ only. Substituting control law (35) into robot dynamics (33), we can write the dynamics of the closed-loop system as

$$\ddot{\theta} = (\mathbf{I} + \mathbf{R}^{-}\Delta\mathbf{R})\mathbf{u} + \mathbf{R}^{-1}\Delta\mathbf{h}, \tag{38}$$

where **I** is the $n \times n$ identity matrix and $\Delta\mathbf{h} = \hat{\mathbf{h}} - \mathbf{h}$ reflects the effects of unmodeled gravitational loads, friction, or model simplification. Define then a set of scalars $\Delta_i = \Delta_i(\theta)$ such that for all θ,

$$\Delta_i \geqslant (\mathbf{L}_i^+)^T\Delta\mathbf{R}_i^+, \quad i = 1, \ldots, n, \tag{39}$$

where the components of $\Delta\mathbf{R}_i^+$ and $\mathbf{L}_i^+$ are the absolute values of the components of $\Delta\mathbf{R}_i$ and $\mathbf{L}_i$. We assume that

$$\Delta_i < \mathbf{L}_i^T\mathbf{R}_i = 1, \quad i = 1, \ldots, n, \tag{40}$$

for all configurations θ. This condition will later be relaxed. The function of multiplier $G_i = G_i(\theta)$ in Eq. (36) is then simply to center the estimated geometric mean gain, according to Section 2.3.1:

$$G_i := ((1 - \Delta_i)(1 + \Delta_i))^{-1/2}, \quad i = 1, \ldots, n. \tag{41}$$

Corresponding gain margins $\beta_i = \beta_i(\theta)$ are

$$\beta_i := \left(\frac{1 + \Delta_i}{1 - \Delta_i}\right)^{1/2}, \quad i = 1, \ldots, n. \tag{42}$$

Gains $k_i(\theta, \dot{\theta}; t)$ are computed so as to meet the simplified conditions

$$k_i(\theta, \dot{\theta}; t) \geq \beta_i((\mathbf{L}_i^+)^T \cdot \sigma(t) + \eta_i), \quad i = 1, \ldots, n, \tag{43}$$

with

$$\sigma(t) := \Delta \mathbf{h}^+ + \sum_{j=1}^{n} |\ddot{\theta}_{dj}| \cdot \Delta \mathbf{R}_j^+, \tag{44}$$

where the η_i in Eq. (43) are the positive constants used in Eq. (4) of each sliding condition and, by definition, the components of $\Delta\mathbf{h}^+$ in Eq. (44) are the absolute values of the components of $\Delta\mathbf{h}$. Conditions (39) and (43) are fairly straightforward to satisfy for $n = 3$ (hence in particular for a six-degree-of-freedom wrist-partitioned robot) as closed-form dynamics remain manageable analytically; in particular, an off-line worst-case analysis may be sufficient in practice (an extreme example of such analysis can be found in Slotine and Sastry 1983). In the general case, an adequate first-order approximation is to evaluate bounds (39) and (43) by using estimates $\hat{\mathbf{L}}_i^+$ (obtained by inverting $\hat{\mathbf{R}}$) in place of the actual $\mathbf{L}_i^+$. Similarly, approximate upper bounds on the components of $\Delta\mathbf{R}_j^+$ or $\Delta\mathbf{h}^+$ may be easily expressed in terms of $\hat{\mathbf{R}}_i^+$ or $\hat{\mathbf{h}}^+$ (with obvious notations); for instance, if the only source of uncertainty on estimated inertia matrix $\hat{\mathbf{R}}$ is that manipulator inertias are known to within a 10 percent precision, one may use in practice the approximation $\Delta\mathbf{R}_i^+ \leq 10\% \; \hat{\mathbf{R}}_i^+$, where the inequality is understood componentwise.

Remarks:

1. Complete expressions of the $k_i(\theta, \dot{\theta}; t)$ are derived in the Appendix.
2. In the case that λ is time-varying (as discussed in Section 2.3.2), the term

$$u_i' := -2\dot{\lambda}(\dot{\tilde{\theta}}_i + \lambda\tilde{\theta}_i)$$

should be added to control law u_i in Eq. (36) while augmenting gain $k_i(\theta, \dot{\theta}; t)$ of Eq. (43) accordingly by the quantity $\beta_i|u_i'|(\beta_i - 1/\beta_i)/(\beta_i + 1/\beta_i)$.

3. Assumption (40) can be relaxed to

$$\mathbf{L}_i^T \cdot \hat{\mathbf{R}}_i > 0, \quad i = 1, \ldots, n. \tag{45}$$

This condition is quite mild: It simply means that u_i contributes to $\ddot{\theta}_i$ with a predictable sign—actually, inequality (45) can always be satisfied by letting $\hat{\mathbf{R}}$ be a (possibly time-varying) positive definite diagonal matrix. In the general case, with

$$0 < \beta_i^{\min} \leq \mathbf{L}_i^T \cdot \hat{\mathbf{R}}_i \leq \beta_i^{\max}, \quad i = 1, \ldots, n, \tag{46}$$

where $\beta_i^{\min} = \beta_i^{\min}(\theta)$ and $\beta_i^{\max} = \beta_i^{\max}(\theta)$, Eqs. (41), (42), and (36) are generalized as

$$G_i := (\beta_i^{\min} \cdot \beta_i^{\max})^{-1/2}, \tag{47}$$

$$\beta_i := (\beta_i^{\max}/\beta_i^{\min})^{+1/2}, \tag{48}$$

$$u_i = -G_i(\theta)[(2/(\beta_i + 1/\beta_i))(-\ddot{\theta}_{di} + 2\lambda\dot{\tilde{\theta}}_i + \lambda^2\tilde{\theta}_i) + \bar{k}_i(\theta, \dot{\theta}; t)\,\mathrm{sat}(s_i/\Phi_i)], \tag{49}$$

with conditions (43) and (44) replaced by

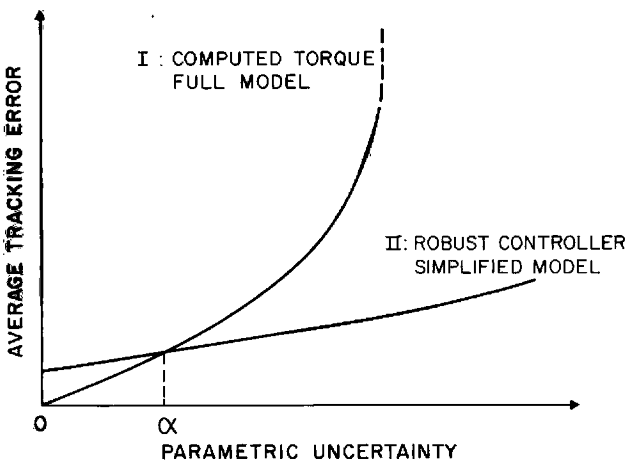
$$k_i(\theta, \dot{\theta}; t) \geq \beta_i\left((\mathbf{L}_i^+)^T\Delta\mathbf{h}^+ + \sum_{j=1}^{n} |\ddot{\theta}_{dj}| \cdot |\mathbf{L}_i^T\hat{\mathbf{R}}_j| \cdot [2G_j|(\beta_j + 1/\beta_j)] + \eta_i\right). \tag{50}$$

The degree of simplification in the system model may be varied according to the on-line computing power available: The balance conditions clearly quantify the trade-off between model precision and tracking accuracy. Further, the s-trajectories provide a measure of the validity of the assumptions on model uncertainty and of the adequacy of bound simplifications.

4. Concluding Remarks

Conceptually, the preceding development can be illustrated by Fig. 5. Consider a fast manipulator motion, say a 1/2-s stop-to-stop trajectory across the workspace, including a full flipping of the wrist, and plot

Fig. 5. Effect of parametric uncertainty on controller performance.



the average tracking precision against parametric uncertainty (say average imprecision on manipulator inertias). Using the computed-torque method based on a full six-degree-of-freedom model, we obtain curve *I*: Tracking is perfect in the absence of model uncertainty and then quickly degrades as uncertainty increases, with the system eventually becoming unstable. Using a robust suction controller based on a simplified model (two "decoupled" three-degree-of-freedom systems), the result is curve II. Because of model simplification, performance is not perfect in the absence of parametric uncertainty, as when using the full six-degree-of-freedom model; but the robust controller quickly outperforms the computed-torque scheme as uncertainty increases. The crossing point α of Fig. 5 typically corresponds to a very low level of parametric uncertainty; further, $\alpha \rightarrow 0$ as allowable bandwidth λ increases, as could be expected from the dynamic balance conditions of Section 2. For direct-drive arms, for instance, the crossing point α of Fig. 5 corresponds to a few percent imprecision on manipulator inertias—a value fairly smaller than typical uncertainty. This effect would likely be further enhanced for geared manipulators involving significant friction terms. The suction control methodology thus combines in practice improved performance with simpler and more tractable controller designs.

Appendix: Uncertainty on the Manipulator Mass Properties

Although simplified conditions (43) or (50) of Section 3.3 were found to be generally adequate in practice, exact expressions for the gains $k_i(\theta, \dot{\theta}; t)$ are presented here for completeness. The derivation is of interest in itself since it provides insight into the effect of uncertainty about the manipulator mass properties. The development is presented directly for the general case of Eq. (46).

Let us define a set of scalars $\Delta_{ij} = \Delta_{ij}(\theta)$ such that, for all θ,

$$\Delta_{ij} \geqslant |\mathbf{L}_i^T \hat{\mathbf{R}}_j|, \quad i = 1, \ldots, n, j = 1, \ldots, n. \quad \text{(A1)}$$

Given the expression of the closed-loop dynamics,

$$\ddot{\theta} = \mathbf{R}^{-1}\hat{\mathbf{R}}\mathbf{u} + \mathbf{R}^{-1}\Delta\mathbf{h},$$

and definitions (47), (48), and (49), gains $k_i(\theta, \dot{\theta}; t)$ must verify

$$k_i(\theta, \dot{\theta}; t) \geqslant \beta_i \left(|u_{i0}|(\beta_i - 1/\beta_i)/(\beta_i + 1/\beta_i) + (\mathbf{L}_i^+)^T \Delta\mathbf{h}^+ + \sum_{j \neq i} \Delta_{ij}|u_j| + \eta_i \right), \quad \text{(A2)}$$

where

$$u_{i0} := -\ddot{\theta}_{di} + 2\lambda\dot{\tilde{\theta}}_i + \lambda^2\tilde{\theta}_i.$$

From the expression (49) of the u_j and the definition (27) of the $\bar{k}_i(\theta, \dot{\theta}; t)$, a sufficient condition for (A2) to be satisfied is that

$$k_i(\theta, \dot{\theta}; t) = k_i'(\theta, \dot{\theta}; t) + \beta_i \sum_{j \neq i} (G_j \Delta_{ij} k_j(\theta, \dot{\theta}; t)|\mathrm{sat}(s_j/\Phi_j)|), \quad \text{(A3)}$$

where $k_i'(\theta, \dot{\theta}; t)$ is defined so that

$$k_i'(\theta, \dot{\theta}; t) \geqslant \beta_i \Bigg(|u_{i0}|(\beta_i - 1/\beta_i)/(\beta_i + 1/\beta_i) + (\mathbf{L}_i^+)^T \Delta\mathbf{h}^+ + \sum_{j \neq i} G_j \Delta_{ij} |2u_{j0}/(\beta_j + 1/\beta_j) + \left(\frac{\lambda\Phi_j}{\beta_j} - k_j(\theta_d, \dot{\theta}_d; t) \right) \mathrm{sat}(s_j/\Phi_j)| + \eta_i \Bigg). \quad \text{(A4)}$$

Now since all terms in $\text{sat}(s_j/\Phi_j)$ vanish for $\theta \equiv \theta_d$, $\dot{\theta} \equiv \dot{\theta}_d$, expressions (A3) and (A4) allow us to compute $k_i(\theta_d, \dot{\theta}_d; t) = k_i'(\theta_d, \dot{\theta}_d; t)$ directly. Further, from the dynamic balance conditions (28)–(30), boundary layer thicknesses Φ_j depend only on the $k_j(\theta_d, \dot{\theta}_d; t)$, so that in turn the knowledge of the $k_j(\theta_d, \dot{\theta}_d; t)$ allows expressions (A3) and (A4) to completely define the $k_i(\theta, \dot{\theta}; t)$: The vector $\mathbf{k}(\theta, \dot{\theta}; t)$ of components $k_i(\theta, \dot{\theta}; t)$ is the solution of the linear system

$$\mathbf{A}(\theta, \dot{\theta}; t)\mathbf{k}(\theta, \dot{\theta}; t) = \mathbf{k}'(\theta, \dot{\theta}; t), \qquad \text{(A5)}$$

where $\mathbf{k}'(\theta, \dot{\theta}; t)$ is the vector of components $k_i'(\theta, \dot{\theta}; t)$, and the matrix $\mathbf{A}(\theta, \dot{\theta}; t)$ is defined as

$$\mathbf{A}(\theta, \dot{\theta}; t) := \begin{bmatrix} 1 & -\beta_1 G_2 \Delta_{12}|\text{sat}(s_2/\Phi_2)| & \cdots \\ -\beta_2 G_1 \Delta_{21}|\text{sat}(s_1/\Phi_1)| & 1 & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix}.$$

Further, all components $k_i(\theta, \dot{\theta}; t)$ of the solution $\mathbf{k}(\theta, \dot{\theta}; t)$ of Eq. (A5) must be strictly positive for this solution to be admissible: Given the Metzler structure (Siljak 1978) of the matrix $-\mathbf{A}$, this is the case as long as all real eigenvalues of $\mathbf{A}$ remain positive. Equivalently, defining $\mathbf{A}_0 = \mathbf{A}_0(\theta, \dot{\theta}; t)$ as

$$\mathbf{A}_0 := \mathbf{diag}(\beta_j^{-1})\mathbf{A}\ \mathbf{diag}(G_j^{-1})$$
$$= \begin{bmatrix} \beta_1^{\min} & -\Delta_{12}|\text{sat}(s_2/\Phi_2)| & \cdots \\ -\Delta_{21}|\text{sat}(s_1/\Phi_1)| & \beta_2^{\min} & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix}, \qquad \text{(A6)}$$

where the second equality stems from definitions (47) and (48) of the G_j and β_j, then $\mathbf{k}(\theta, \dot{\theta}; t)$ of Eq. (A5) is admissible as long as all real eigenvalues of $\mathbf{A}_0$ remain positive. Note that $\mathbf{A} \equiv \mathbf{A}_0 \equiv \mathbf{I}$ in the absence of uncertainty on the inertia matrix.

Remarks:

1. A sufficient condition for $\mathbf{k}(\theta, \dot{\theta}; t)$ to be admissible is that all real eigenvalues of the matrix

$$\mathbf{A}_0' := \begin{bmatrix} \beta_1^{\min} & -\Delta_{12} & \cdots \\ -\Delta_{21} & \beta_2^{\min} & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix}$$

be positive.

2. Regardless of the level of parametric uncertainty, it is always possible to generate admissible $\mathbf{k}(\theta, \dot{\theta}; t)$ by selecting large enough $\mathbf{k}'(\theta, \dot{\theta}; t)$. This can be achieved by multiplying the right-hand side of inequality (A4) by an appropriately large scaling factor ρ, such as a constant upper bound on the Frobenius-Perron roof[3] ρ_1 of the matrix

$$\mathbf{A}_1 := \begin{bmatrix} 0 & \beta_1 G_2 \Delta_{12} & \cdots \\ \beta_2 G_1 \Delta_{21} & 0 & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix}.$$

We assume that $\rho_1 > 1$, or else gain vector $\mathbf{k}(\theta, \dot{\theta}; t)$ is already guaranteed to be admissible. This scaling has the effect of artificially increasing boundary layer thicknesses Φ_j by the same factor ρ without actually modifying the control action inside the (original and still effective) boundary layers B_j^0 since gains $\bar{k}_j(\theta, \dot{\theta}; t)$ also scale up by the same factor. The values of the s_j are thus left unchanged (as is the tracking performance, which allows us to use constant ρ's), so that the off-diagonal terms of the matrix $\mathbf{A}$ are divided by the scaling fac-

3. See, for example, Siljak (1978). The Frobenius-Perron roof ρ_1 of a matrix $\mathbf{A}_1$ with nonnegative elements is the largest (automatically nonnegative) real eigenvalue of $\mathbf{A}_1$. The equation $(\mathbf{I} - \rho^{-1}\mathbf{A}_1)\mathbf{y} = \mathbf{y}'$ admits positive solutions $\mathbf{y}$ for positive $\mathbf{y}'$ if $\rho > \rho_1$.

tor, which in turn guarantees that the solution $\mathbf{k}(\theta, \dot{\theta}; t)$ of Eq. (A5) is admissible. Further, it is desirable that (A5) be guaranteed to have admissible solutions regardless of the values of the s_j. This can be achieved by substituting $\rho_I^{-1}\text{sat}(\rho_I s_j/\Phi_j)$ to $\text{sat}(s_j/\Phi_j)$ in the expression of $\mathbf{A}$, with $\rho_1 \leq \rho_I \leq \rho$, which further guarantees that the B_j^0 are attractive for all trajectories starting inside boundary layers of thicknesses $(\rho/\rho_I)\Phi_j^0$ (where Φ_j^0 is the thickness of B_j^0). Note that a convenient upper bound of ρ_1 is

$$\rho_1 \leq \min\left(\max_i(G_i \sum_{j \neq i} \Delta_{ji}\beta_j), \max_i(\beta_i \sum_{j \neq i} \Delta_{ij}G_j)\right).$$

3. The solution $\mathbf{k}(\theta, \dot{\theta}; t)$ of Eq. (A5) is bounded for bounded $\boldsymbol{\theta}_d$, provided all real eigenvalues of $\mathbf{A}_0$ (or $\mathbf{A}$) are uniformly bounded away from zero (that is, remain larger than some strictly positive constant). Indeed, $\mathbf{A}^{-1}(\theta, \dot{\theta}; t)$ is then bounded for bounded θ, $\dot{\theta}$, and, further, $\mathbf{k}(\theta, \dot{\theta}; t)$ is admissible. This in turn implies that tracking error is indeed limited by boundary layer thicknesses Φ_j, which depend only on the $k_j(\theta_d, \dot{\theta}_d; t)$, so that θ, $\dot{\theta}$ and thus both $\mathbf{A}^{-1}(\theta, \dot{\theta}; t)$ and $\mathbf{k}'(\theta, \dot{\theta}; t)$ are bounded.
4. The condition that $k(\theta, \dot{\theta}; t)$ of Eq. (A5) be admissible can be given a slightly different interpretation (based again on the Metzler structure of $-\mathbf{A}_0$). It is satisfied if

$$\det \begin{bmatrix} \beta_1^{\min} & -\Delta_{12}\chi_2 & \cdots \\ -\Delta_{21}\chi_1 & \beta_2^{\min} & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix} > 0, \quad \text{(A7)}$$

for all χ_j such that $0 \leq \chi_j \leq |\text{sat}(s_j/\Phi_j)|$. Let us now assume that

$$\det \hat{\mathbf{R}} > 0. \quad \text{(A8)}$$

Condition (A8) is automatically satisfied for any physically motivated choice of $\hat{\mathbf{R}}$, since then $\hat{\mathbf{R}} = \hat{\mathbf{R}}^T > 0$. Now from $\mathbf{R} = \mathbf{R}^T > 0$, condition (A8) implies that

$$\det(\mathbf{R}^{-1}\hat{\mathbf{R}}) > 0, \quad \text{(A9)}$$

which can be written

$$\det \begin{bmatrix} \mathbf{L}_1^T\hat{\mathbf{R}}_1 & \mathbf{L}_1^T\hat{\mathbf{R}}_2 & \cdots \\ \mathbf{L}_2^T\hat{\mathbf{R}}_1 & \mathbf{L}_2^T\hat{\mathbf{R}}_2 & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix} > 0. \quad \text{(A10)}$$

The comparison of expressions (A7) and (A10) shows that it is desirable that the $\beta_i^{\min}$ of Eq. (46) and the Δ_{ij} of Eq. (A1) be compatible, that is, be generated in a way that preserves the natural structure of Eq. (A10).

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