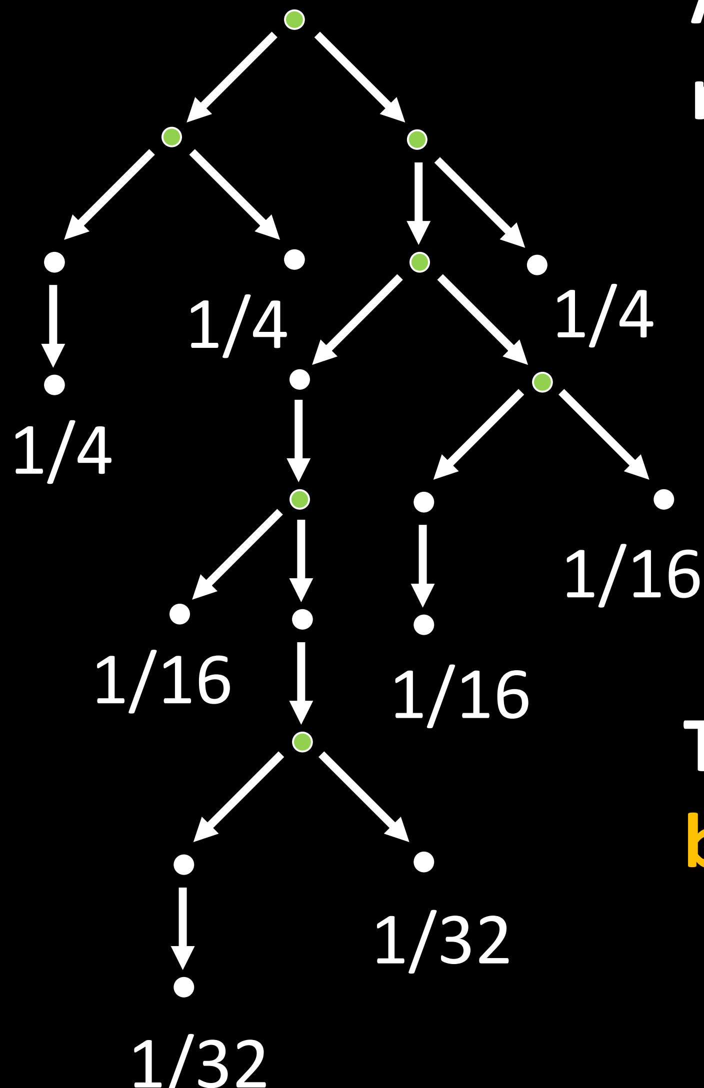
6.045

Lecture 22: Finish Randomized Complexity, Summary of 6.045

Randomized / Probabilistic Complexity

Probabilistic TMs



A **probabilistic TM** M is a nondeterministic TM where:

- Each nondeterministic step is called a **coin flip**
- Each nondeterministic step has only two legal next moves **(heads or tails)**

The probability that M runs on a **branch b** is: $\Pr[\, b \,] = 2^{-k}$

where k is the number of coin flips that occur on branch b

Definition. A probabilistic TM M decides a language A with error ε if for all strings w,

$$w \in A \Rightarrow \Pr[\, M \text{ accepts } w \,] \geq 1 - \varepsilon$$

$$w \notin A \Rightarrow \Pr[\, M \text{ doesn't accept } w \,] \geq 1 - \varepsilon$$

Theorem: A language A is in NP if there is a nondeterministic polynomial time TM M such that for all strings w:

$$w \in A \Rightarrow \Pr[\, M \text{ accepts } w \,] > 0$$

$$w \notin A \Rightarrow \Pr[\, M \text{ accepts } w \,] = 0$$

BPP = Bounded Probabilistic P

BPP = { L | L is recognized by a probabilistic polynomial-time TM with error at most **1/3** }

Why 1/3?

It doesn't matter what error value we pick, as long as the error is smaller than 1/2.

When the error is smaller than 1/2, we can make it very small by repeatedly running the TM.

An arithmetic formula is like a Boolean formula, except it has +, –, and * instead of OR, NOT, AND.

ZERO-POLY = { p | p is an arithmetic formula over Z that is *identically* zero}

Identically zero means: all coefficients are 0

Two examples of formulas in ZERO-POLY:

$(x + y)\cdot(x + y) - x\cdot x - y\cdot y - 2\cdot x\cdot y$

Abbreviate as: $(x + y)^2 - x^2 - y^2 - 2xy$

$(x^2 + a^2)\cdot(y^2 + b^2) - (x\cdot y - a\cdot b)^2 - (x\cdot b + a\cdot y)^2$

There is a rich history of polynomial identities in mathematics. Useful also in program testing!

Testing Univariate Polynomials

Let p(x) be a polynomial in one variable over Z

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_dx^d$$

Suppose p is hidden in a "black box" –
we can only see its inputs and outputs.
Want to determine if p is *identically* 0

Simply evaluate p on d+1 distinct values!
Non-zero degree d polynomials have ≤ d roots.
But the *zero polynomial* has every value as a root.

Testing Multivariate Polynomials

Let $p(x_1,\ldots,x_n)$ be a polynomial in n variables over Z

Suppose $p(x_1,\ldots,x_n)$ is given to us, but as a very complicated arithmetic formula.

Can we *efficiently* determine if p is identically 0?

If $p(x_1,\ldots,x_n)$ is a product of m polynomials, each of which is a polynomial in t terms, $\prod_m(\sum_t stuff)$

Then expanding the expression into a $\sum$ of $\prod$ could take t^m time!

Big Idea: Evaluate p on *random values*

Theorem (Schwartz-Zippel-DeMillo-Lipton)

Let $p(x_1,x_2,\ldots,x_n)$ be a *nonzero* polynomial, where each x_i has degree at most d. Let $F \subset Z$ be finite.

If $a_1,\ldots, a_m$ are selected randomly from F, then:

$$\Pr [\ p(a_1, \ldots, a_m) = 0\ ] \leq dn/|F|$$

Low-deg. nonzero polynomials are nonzero on MANY inputs

Proof (by induction on n):

Base Case (n = 1):

$$\Pr [\ p(a_1) = 0\ ] \leq d/|F|$$

Nonzero polynomials of degree d have most d roots, so at most d elements in F can make p zero

Inductive Step ($n > 1$): Assume true for $n-1$ and prove for n

Let $p(x_1,\ldots,x_n)$ be not identically zero.

Write: $p(x_1,\ldots,x_n) = p_0 + x_n p_1 + x_n^2 p_2 + \ldots + x_n^d p_d$

where x_n does not occur in any $p_i(x_1,\ldots,x_{n-1})$

Observe: At least one p_i is not identically zero

Suppose $p(a_1,\ldots,a_n) = 0$. Let $q(x_n) = p(a_1,\ldots,a_{n-1},x_n)$. Two cases:

(1) $q \equiv 0$. That is, for *all* j, $p_j(a_1,\ldots,a_{n-1}) = 0$ (including p_i)

$\Pr[\,(1)\,] \leq \Pr[p_i(a_1,\ldots,a_{n-1}) = 0] \leq (n-1)d/|F|$ by induction

(2) q is not identically zero, but $q(a_n) = 0$.

Note q is a univariate degree-d polynomial!

$\Pr[\,(2)\,] \leq \Pr[q(a_n) = 0] \leq d/|F|$ by univariate case

$\Pr[\,(1) \text{ or } (2)\,] \leq \Pr[(1)] + \Pr[(2)] \leq nd/|F|$

ZERO-POLY = { p | p is an arithmetic formula over Z that is *identically* zero}

Theorem: ZERO-POLY $\in$ BPP

Proof: Suppose $n=|p|$. Then p has $k \le n$ variables, and the *degree* of each variable is at most n.

Algorithm A: Given polynomial p,
For all i = 1,…,k, choose r_i randomly from $\{1,\ldots,3n^2\}$
If $p(r_1, \ldots, r_k) = 0$ then output *zero*
else output *nonzero*

Observe A runs in polynomial time.

If $p \equiv 0$, then Pr[A(p) outputs *zero*] = 1

If $p \not\equiv 0$, then by the Schwartz-Zippel lemma,

$$\Pr[A(p) \text{ outputs } zero] = \Pr_r[p(r) = 0] \leq n^2/3n^2 \leq 1/3$$

Checking Equivalence of Arithmetic Formulas

ZERO-POLY = { p | p is an arithmetic formula that is identically zero}

Theorem: ZERO-POLY $\in$ BPP

EQUIV-POLY = { (p,q) | p and q are arithmetic formulas computing the same polynomial}

Corollary: EQUIV-POLY $\in$ BPP

Proof: (p,q) in EQUIV-POLY $\Leftrightarrow$ p-q in ZERO-POLY
Therefore EQUIV-POLY $\leq_P$ ZERO-POLY
and we get a BPP algorithm for EQUIV-POLY.

See Sipser 10.2 for an application to testing equivalence of simple programs!

Equivalence of Arithmetic Formulas

EQUIV-POLY = { (p,q) | p and q are arithmetic formulas computing the same polynomial}

Corollary: EQUIV-POLY $\in$ BPP

There is a big contrast with Boolean formulas!

EQUIV = { (ϕ,ψ) | ϕ and ψ are Boolean formulas computing the same function}

We showed EQUIV is in coNP. It's also coNP-complete!
TAUTOLOGY $\leq_P$ EQUIV: map ϕ to $(\phi$, T)

ZERO-POLY = { p | p is an arithmetic formula that is identically zero}

Theorem: ZERO-POLY $\in$ BPP

It is not known how to solve ZERO-POLY efficiently *without* randomness!

Thm [KI'04, AvM'11] If ZERO-POLY $\in$ P then NEW *LOWER BOUNDS FOLLOW*
(not P ≠ NP, but still breakthroughs!)

BPP = { L | L is recognized by a probabilistic polynomial-time TM with error at most 1/3 }

Is BPP $\subseteq$ NP?

THIS IS AN OPEN QUESTION!

Is BPP $\subseteq$ PSPACE?

Yes! Run through all possible sequences of coin flips one at a time, and count the number of branches that accept.

Known: BPP $\subseteq$ NP^{NP} and BPP $\subseteq$ $coNP^{NP}$, but BPP $\subseteq$ P^{NP} is still open!

Is NP $\subseteq$ BPP?

THIS IS AN OPEN QUESTION!

Is BPP = EXPTIME?

THIS IS AN OPEN QUESTION!?*!#!

It’s widely conjectured that P = BPP!

Certain lower bounds $\Rightarrow$ P = BPP

Is BPP = EXPTIME?

THIS IS AN OPEN QUESTION!?*!#!

It’s widely conjectured that P = BPP!

Certain lower bounds ⇒ P = BPP

Definition: A language A is in **RP (Randomized P)** if there is a nondeterministic polynomial time TM M such that for all strings x:

$$x \notin A \Rightarrow \Pr[M(x) \text{ accepts}] = 0$$

$$x \in A \Rightarrow \Pr[M(x) \text{ accepts}] > 2/3$$

NONZERO-POLY = { p | p is an arithmetic formula that is not identically zero}

Theorem: NONZERO-POLY $\in$ RP
(Our proof of ZERO-POLY in BPP shows this)

Is RP $\subseteq$ NP?

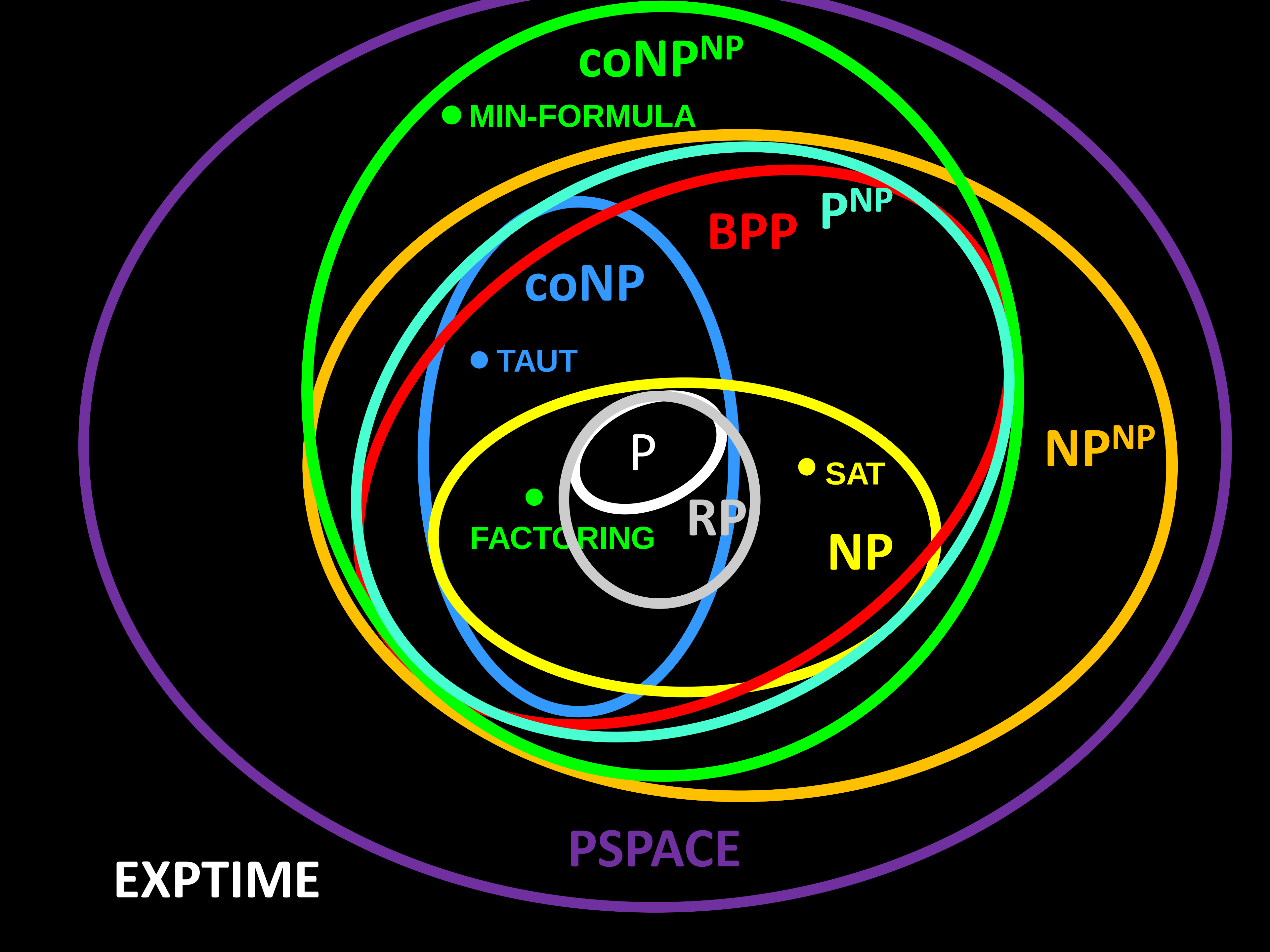
Yes!
Being RP means that not only are there "nifty proofs" but in fact most strings are nifty proofs!

Is RP $\subseteq$ BPP?

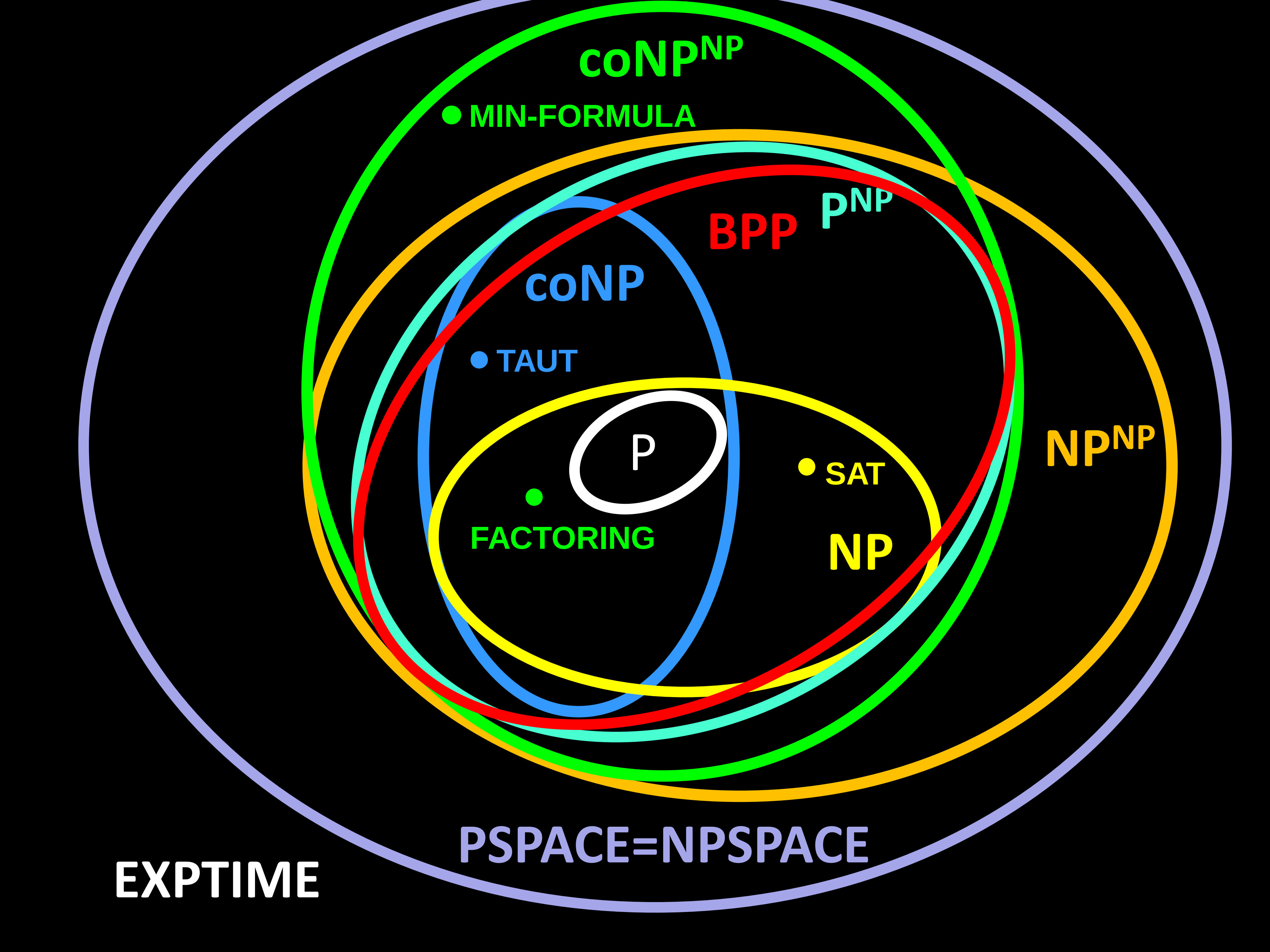
Yes!

RP has "one-sided error"

BPP has "two-sided error"



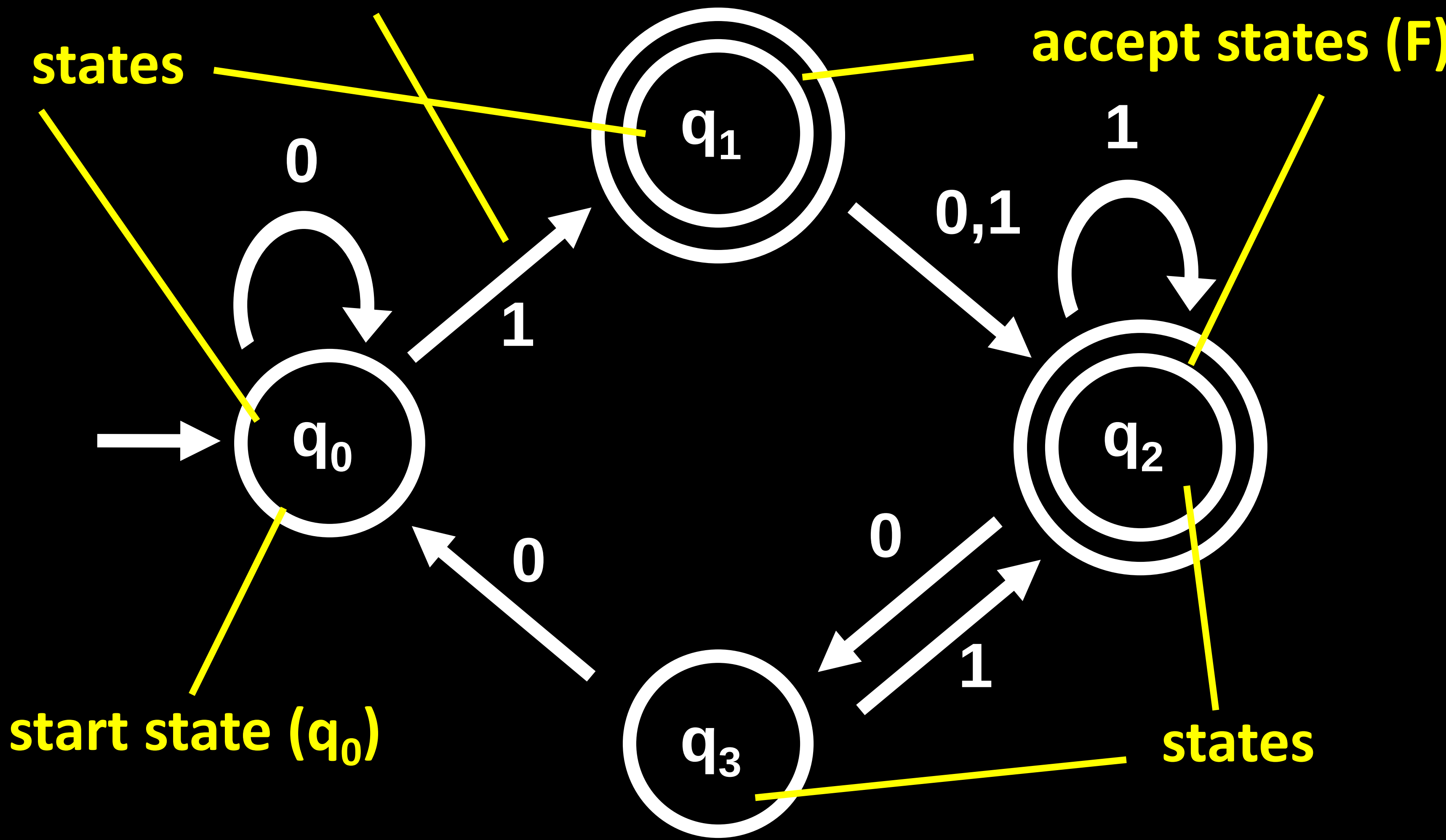
coNP^NP
MIN-FORMULA
P^NP
BPP
coNP
TAUT
P
SAT
RP
FACTORING
NP
NP^NP
PSPACE
EXPTIME


coNP^NP
MIN-FORMULA
P^NP
BPP
coNP
TAUT
P
SAT
NP^NP
FACTORING
NP
PSPACE=NPSPACE
EXPTIME

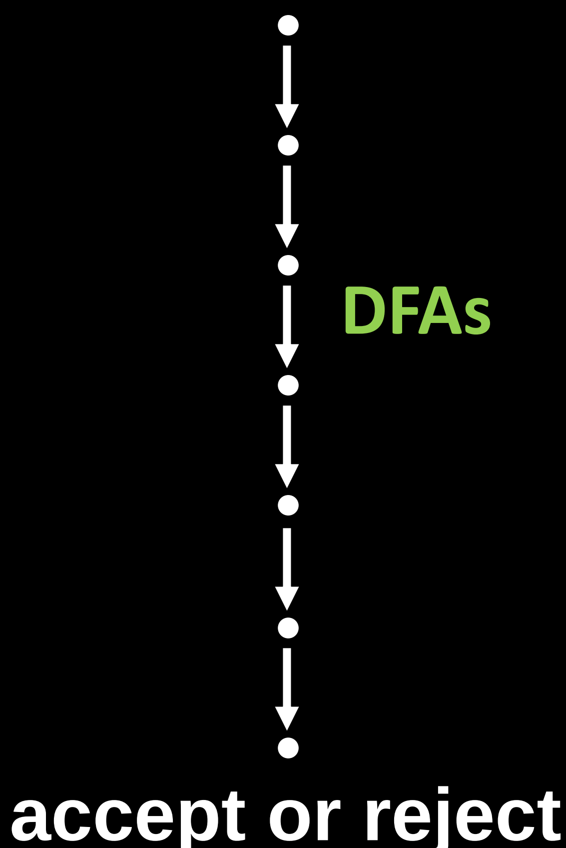
Review

Deterministic Finite Automata

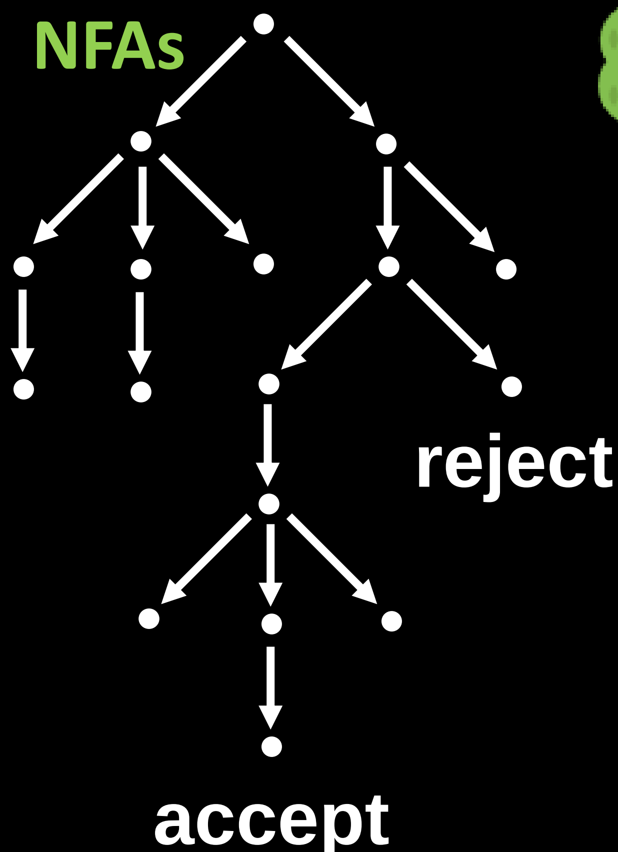
transition: ***for every state and alphabet symbol***



Deterministic Computation

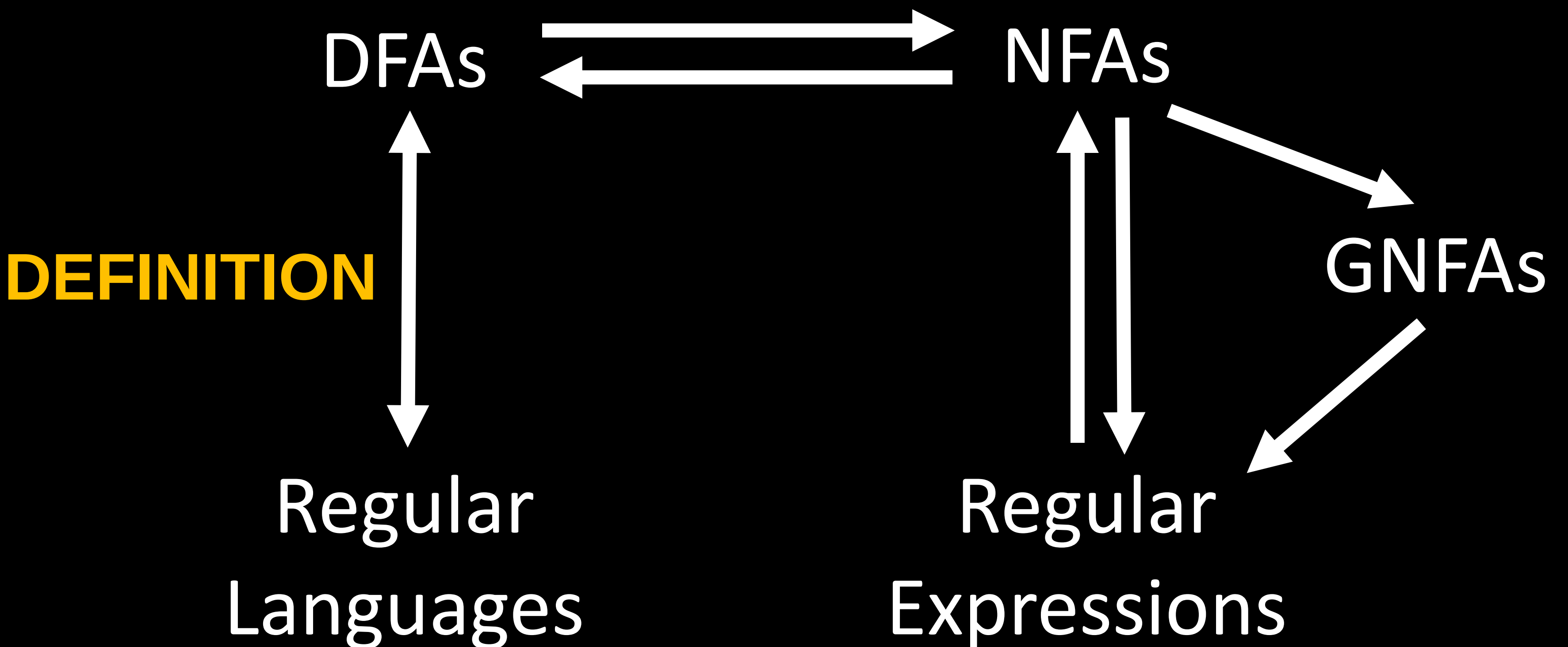


Non-Deterministic Computation



Are these equally powerful???

YES for finite automata


DFAs
NFAs
DEFINITION
GNFAs
Regular Languages
Regular Expressions

Regular Languages are closed under all of the following operations:

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Complement: $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

L is regular

if and only if

$(\exists$ DFA M$)(\forall$ strings $x)$[M acc. x $\Leftrightarrow$ x $\in$ L]

"M gives the correct output on all strings"

L is NOT regular

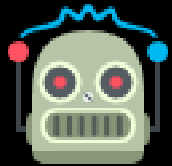
if and only if

$(\forall$ DFA M$)(\exists$ string $x_M)$[M acc. x_M $\Leftrightarrow$ x $\notin$ L]

"M gives the wrong output on x_M"

So the problem of proving L is NOT regular can be viewed as a problem about designing "bad inputs"

How to Confuse DFAs



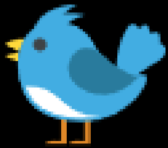
Want to show: Language L is not regular

Proof: By contradiction. Assume L *is* regular.

So L has a DFA M with Q states, for some Q > 0.

YOU: Cleverly pick strings x, y where |y| > Q

Run M on xy. *Pigeons tell us: Some state q of M is visited more than once, while reading in y.*



Therefore, M is in state q after reading xy', and is in q after reading xy'', for distinct prefixes y' and y'' of y

YOU: Cleverly pick string z so that *exactly one* of xy'z and xy''z is in L

But M will give the same output on both! *Contradiction!*

DFA Minimization:

There is an *efficient algorithm* which, given any DFA M, will output the unique minimum-state DFA M* equivalent to M.

If this were true for more general models of computation, that would be an engineering breakthrough!!
(Would imply P=NP, for example)

Table-Filling Algorithm
to find "distinguishable" pairs of states

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ iff for all $z \in \Sigma^*$, $[xz \in L \Leftrightarrow yz \in L]$

The Myhill-Nerode Theorem:
A language L is regular *if and only if* the number of equivalence classes of $\equiv_L$ is *finite*.

Regular = "easy"

Not Regular = "hard"

The **Myhill-Nerode Theorem** gives us a (universal) way to prove that a given language is not regular:

L is not regular

if and only if

there are infinitely many equiv. classes of $\equiv_L$

Distinguishing set for L

L is not regular

if and only if

There are infinitely many strings $w_1, w_2, \ldots$ so that

for all $w_i \neq w_j$, *w_i and w_j are distinguishable to L:*

there is a $z \in \Sigma^*$ such that

exactly one of $w_i z$ and $w_j z$ is in L

Streaming Algorithms

Have three components:

Initialize:

<variables and their assignments>

When next symbol seen is σ:

<pseudocode using σ and vars>

When stream stops (end of string):

<*accept/reject* condition on vars>

(or: <pseudocode for output>)

Algorithm A computes $L \subseteq \Sigma^\star$ if
A accepts the strings in L, rejects strings not in L

L = {x | x has odd number of 1's}

Has streaming algorithms using O(1) space
(that is, it has a DFA) "very easy"

L = {x | x has more 1's than 0's}

Has streaming algorithms using O(log n) space,
no streaming algorithm uses much less "easy"

L = {x | x is a palindrome}

Has streaming algorithms using O(n) space,
no streaming algorithm uses much less "hard"

For any $L \subseteq \Sigma^*$ define $L_n = \{x \in L \mid |x| \leq n\}$

A **streaming distinguisher for** L_n is a subset D_n of Σ^*: for all distinct $x, y \in D_n$, there is a z in Σ^* such that $|xz| \leq n$, $|yz| \leq n$, and *exactly one* of xz, yz is in L.

Streaming Theorem: Suppose for all n, there is a streaming distinguisher D_n for L_n with $|D_n| \geq 2^{S(n)}$. Then all streaming algs for L must use at least $S(n)$ space!

Idea: Use the set D_n to show that every streaming algorithm for L must enter at least $2^{S(n)}$ different memory states, over all inputs of length at most n.

But if there are at least $2^{S(n)}$ distinct memory states, Then the alg must be using at least $S(n)$ bits of space!

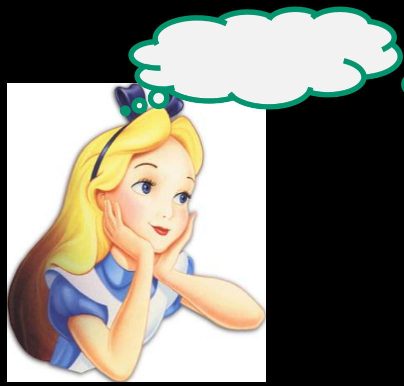
Communication Complexity

A theoretical model of distributed computing

- **Function $f: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$**
  - Two inputs, $x \in \{0,1\}^*$ and $y \in \{0,1\}^*$
  - **We assume $|x|=|y|=n$. Think of n as HUGE**
- **Two computers: Alice** and **Bob**
  - **Alice** *only* knows x, **Bob** *only* knows y
- **Goal: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob**

We do not count computation cost. We *only* care about the number of bits communicated.

Connection to Streaming and DFAs




x y

Let $L \subseteq \{0,1\}^*$

Def. f_L: $\{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$

for x, y with $|x|=|y|$ as:

$$f_L(x, y) = 1 \Leftrightarrow xy \in L$$

Theorem: If L has a streaming algorithm using $\leq s$ space, then cc(f_L) is at most $2s + 1$.

Lower bounds on cc

➔ Lower bounds on streaming
(even with multiple passes)

Connection to Streaming and DFAs

x **y**

Let $L \subseteq \{0,1\}^*$

Def. $f_L: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$

for x, y with $|x|=|y|$ as:

$$f_L(x, y) = 1 \Leftrightarrow xy \in L$$

Examples:

L = { x | x has an odd number of 1s}

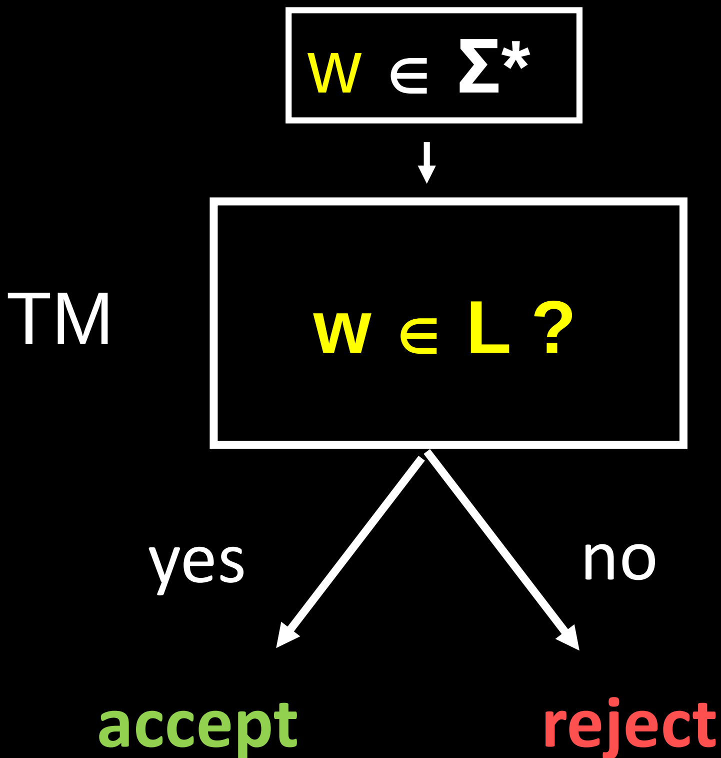
$\Rightarrow f_L(x, y)$ = PARITY(x,y) has $\Theta(1)$ comm. compl.

L = { x | x has more 1s than 0s}

$\Rightarrow f_L(x, y)$ = MAJORITY(x,y) has $\Theta(\log n)$ comm. compl.

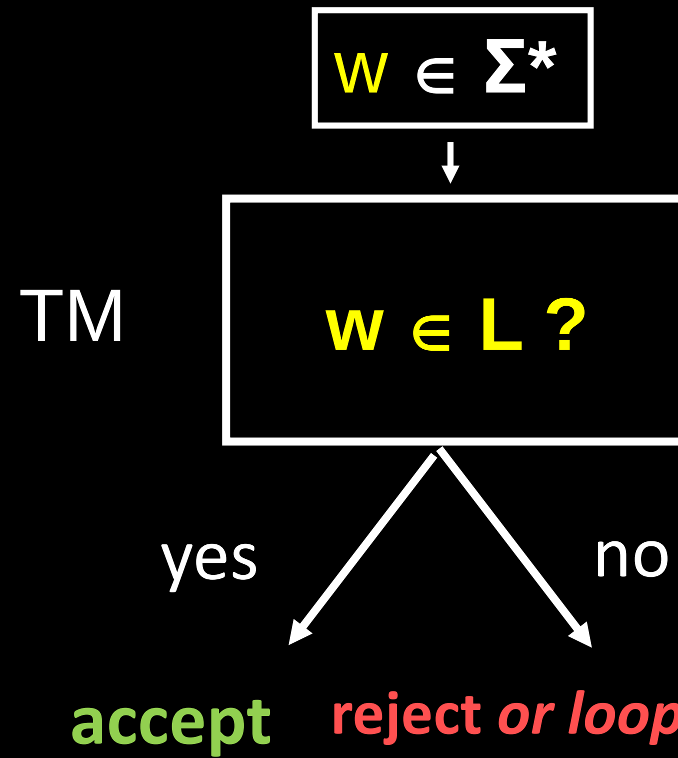
L = { xx | x $\in$ {0,1}*}

$\Rightarrow f_L(x, y)$ = EQUALS(x,y) has $\Theta(n)$ comm. compl.



L is decidable

“easy”



L is recognizable

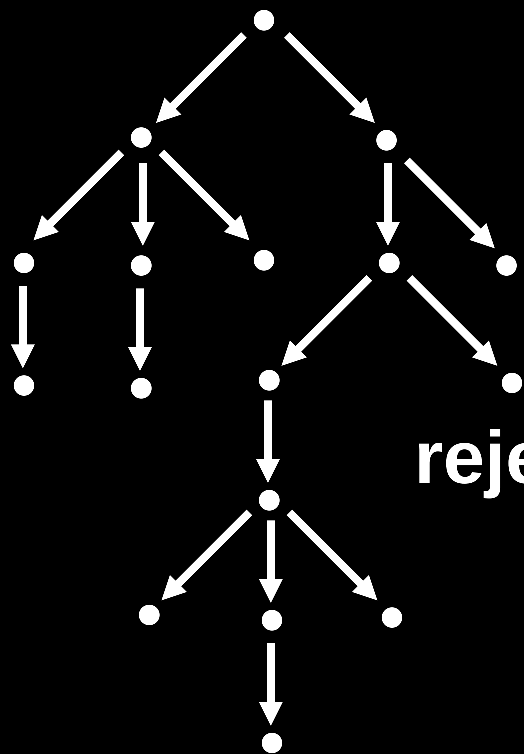
“not so easy”

Theorem: L is decidable

iff both L and ¬L are recognizable

Decidable Computation | **Recognizable Computation**

accept or reject

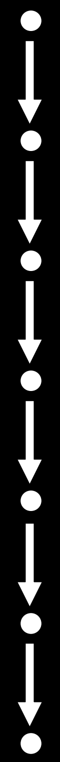


accept

Theorem: L is **recognizable** $\Leftrightarrow$ There is a TM V halting on all inputs such that

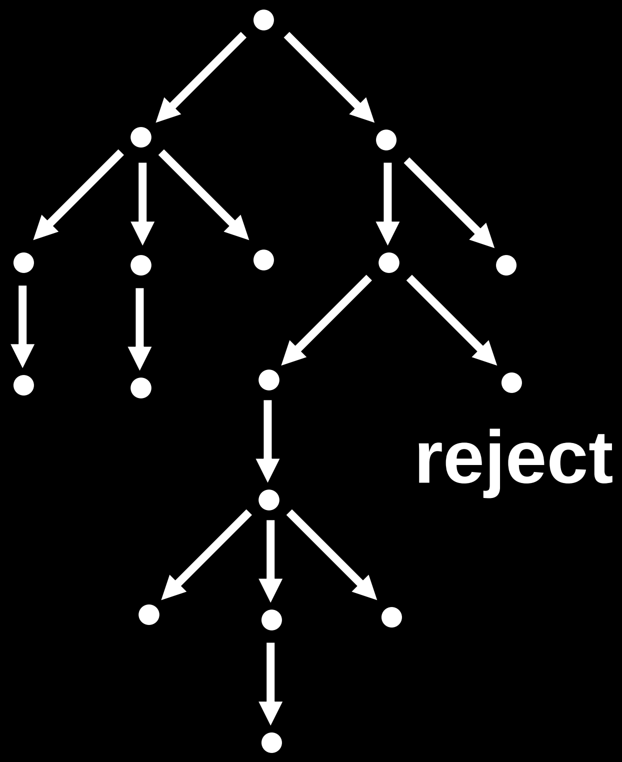
$$L = \{ x \mid \exists y \in \Sigma^* [V(x,y) \text{ accepts }] \}$$

Decidable Computation



accept or reject

Recognizable Computation



accept

Are these equally powerful???

NO for Turing Machines

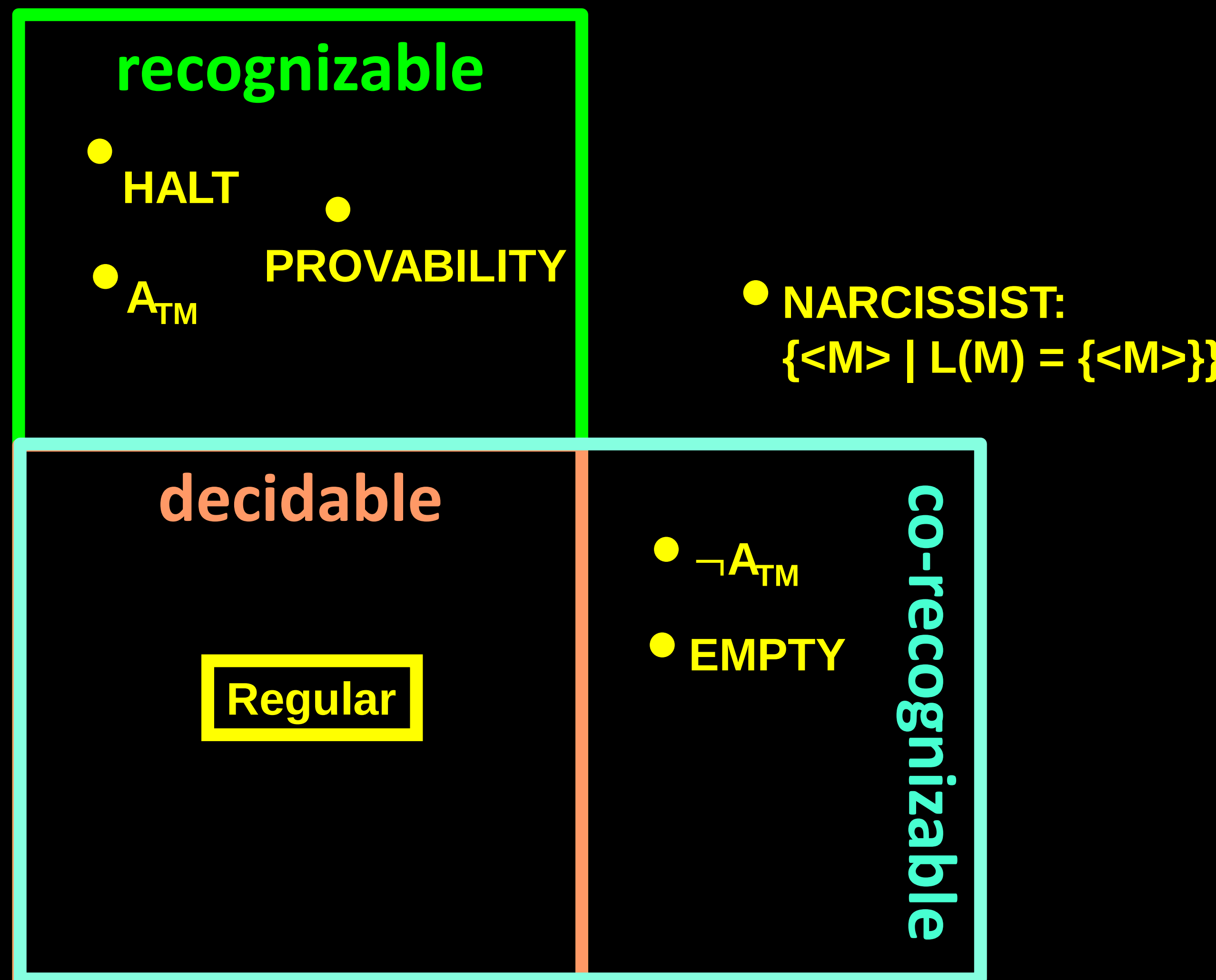
Diagonalization

Mapping Reductions and Oracle Reductions

Rice's Theorem

Recursion Theorem

Gödel's Theorems



Decidable = Recognizable ∩ Co-recognizable

Church-Turing Thesis

A_{TM} = { ⟨M,w⟩ | M is a TM that accepts string w }

Thm. A_{TM} is undecidable: (proof by contradiction)

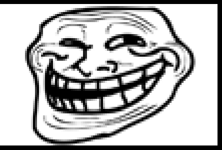
Assume H is a machine that decides A_{TM}

$$H(\langle M,w\rangle) = \begin{cases} \text{Accept} & \text{if } M \text{ accepts } w \\ \text{Reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Define a new TM D with the following spec:

D(⟨M⟩): Run H on ⟨M,M⟩ and output the *opposite* of H

$$D(\langle D\rangle) = \begin{cases} \text{Reject} & \text{if } D \text{ accepts } \langle D\rangle \\ \text{Accept} & \text{if } D \text{ does not accept } \langle D\rangle \end{cases}$$



Set M=D?

A_{TM} = { ⟨M,w⟩ | M is a TM that accepts string w }

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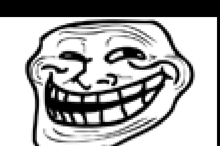

Set M=D?

Mapping Reductions

$f : \Sigma^* \rightarrow \Sigma^*$ is a **computable function** if
there is a Turing machine M that halts with
just $f(w)$ written on its tape, for every input w

A language A is *mapping reducible* to language B,
written as $A \leq_m B$, if there is a computable
$f : \Sigma^* \rightarrow \Sigma^*$ such that for every $w \in \Sigma^*$,

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a mapping reduction
(or many-one reduction) from A to B

Recursion Thm: For every computable t, there is a computable r such that r(w) = t(R,w) where *R is a description of a TM computing r*

Moral: Suppose we can design a TM T of the form
"On input (x,w), do bla bla with x,
do bla bla bla with w, etc. etc."
We can always find a TM R with the *behavior*:
"On input w, do bla bla with code of R,
do bla bla bla with w, etc. etc."

We can use the operation:
"Obtain your own description" 
in Turing machine pseudocode!

Limitations on Mathematics

For every consistent and interesting $\mathcal{F}$,

Theorem 1. (Gödel 1931) $\mathcal{F}$ is *incomplete:*
There are mathematical statements in $\mathcal{F}$ that are *true* but cannot be proved in $\mathcal{F}$.

Theorem 2. (Gödel 1931) The consistency of $\mathcal{F}$ cannot be proved in $\mathcal{F}$.

Theorem 3. (Church-Turing 1936) The problem of checking whether a given statement in $\mathcal{F}$ has a proof is undecidable.

😱 🚫 Limitations on Mathematics

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Theorem 3. (Church-Turing 1936) The problem of checking whether a given statement in $\mathcal{F}$ has a proof is undecidable.

Time Complexity

Definition:

TIME(t(n)) = { L′ | there is a Turing machine M with time complexity O(t(n)) so that L′ = L(M) }

= { L′ | L′ is a language decided by a Turing machine with $\leq$ c t(n) + c running time }

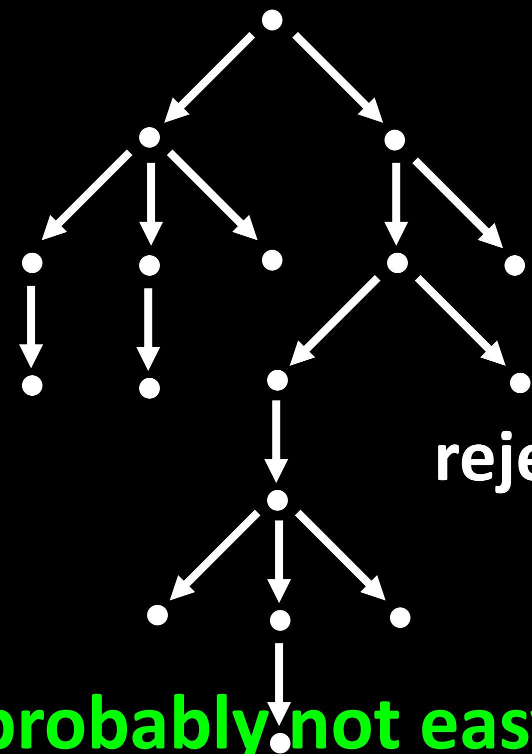
The Time Hierarchy Theorem

Intuition: **The more computing time you have, the more problems you can solve.**

Theorem: **For all "reasonable" f, g : $\mathbb{N} \to \mathbb{N}$ where for all n, $g(n) > n^2 f(n)^2$,** $\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$

Deterministic
Poly-Time Computation

Non-Deterministic
Poly-Time Computation



Are these equally powerful???

P = NP ????

Theorem: $L \in NP \Leftrightarrow$ There is a constant k and polynomial-time TM V such that

$$L = \{ x \mid \exists y \in \Sigma^* [\, |y| \leq |x|^k \text{ and } V(x,y) \text{ accepts} \,] \}$$

Moral: A language L is in NP
if and only if
there are polynomial-length ("nifty") proofs
for membership in L

Theorem: L is recognizable $\Leftrightarrow$
There is a TM V that halts on all inputs such that

$$L = \{ x \mid \exists y \in \Sigma^* [\, V(x,y) \text{ accepts} \,] \}$$

Definition: A language B is NP-complete if:

1. $B \in NP$
2. Every A in NP is poly-time reducible to B

   That is, $A \leq_P B$

   When this is true, we say "B is NP-hard"

NP-complete problems: "very likely hard"

3SAT, SAT, CLIQUE, IS, VC, SUBSET-SUM, KNAPSACK, PARTITION, BIN-PACKING, …

Definition: coNP = $\{ L \mid \neg L \in NP \}$

What does a coNP problem L look like?

The instances not in L have *nifty proofs*.

Any NP problem L can be written in the form:

L = {x | ∃y of poly(|x|) length so that V(x,y) accepts}

¬L = {x | ¬∃y of poly(|x|) length so that V(x,y) accepts}

= {x | ∀y of poly(|x|) length, V(x,y) rejects}

Instead of using an "existentially guessing" (nondeterministic) machine,

we can define a "universally verifying" machine!

Complexity Classes With Oracles

Let B be a language.

B

P 

P^B = { L | L can be decided by some *polynomial-time* TM with an oracle for B }

P^{NP} = the class of languages decidable by *some* polynomial-time oracle TM with an oracle for *some* B in NP

NP^{NP} = the class of languages decidable by *some* nondeterministic polynomial-time oracle TM with an oracle for *some* B in NP

NP-complete problems:

NHALT, SAT, 3SAT, CLIQUE, VC, SUBSET-SUM, …

coNP-complete problems:

UNSAT, TAUTOLOGY, NOHAMPATH, …

PSPACE-complete problems:

SPACE-HALT, TQBF, GG

There are also NP^{NP}-complete and $coNP^{NP}$ problems
(but you don’t need to know them for the final!)

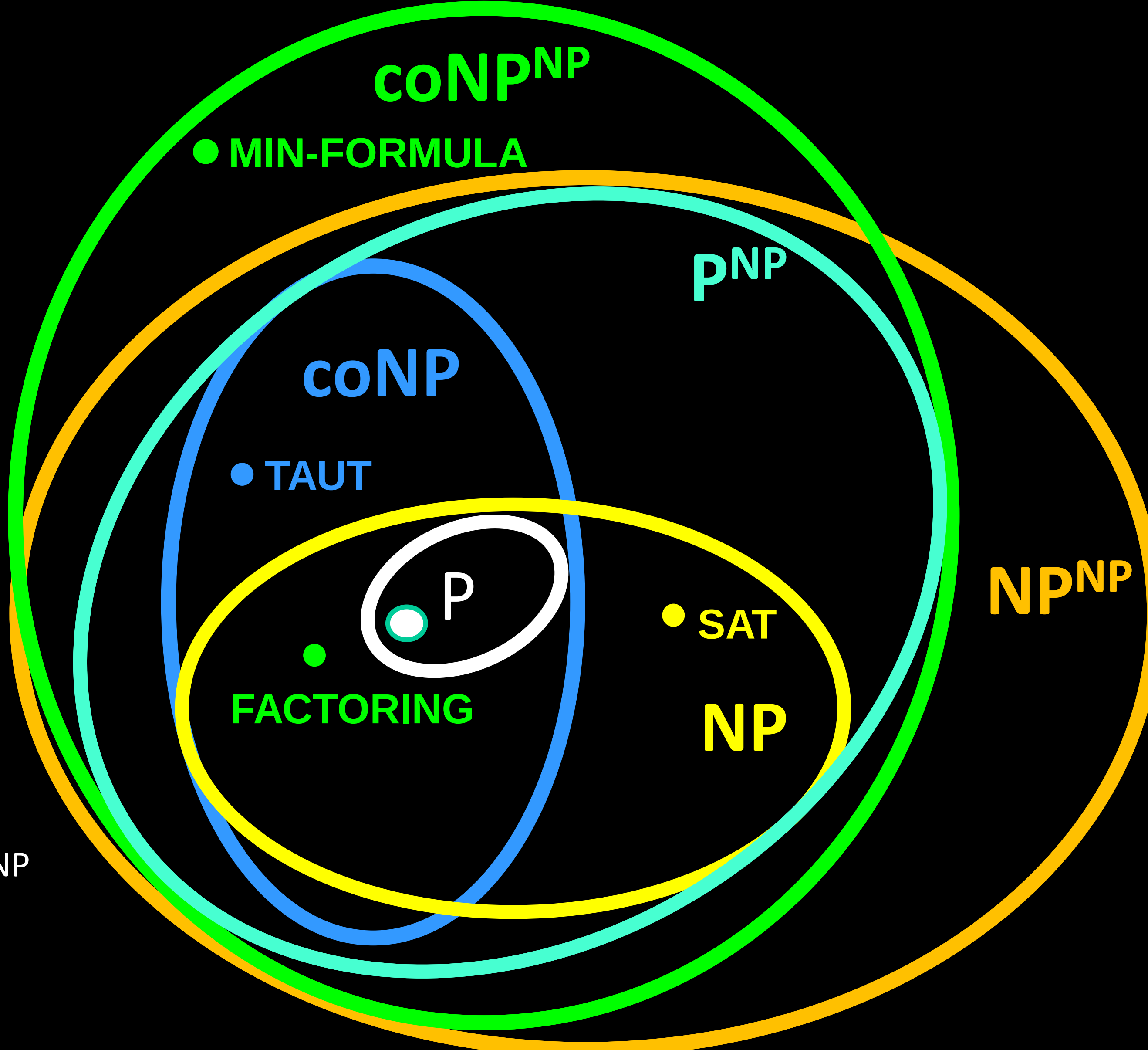
Time Hierarchy

Poly-time Reductions

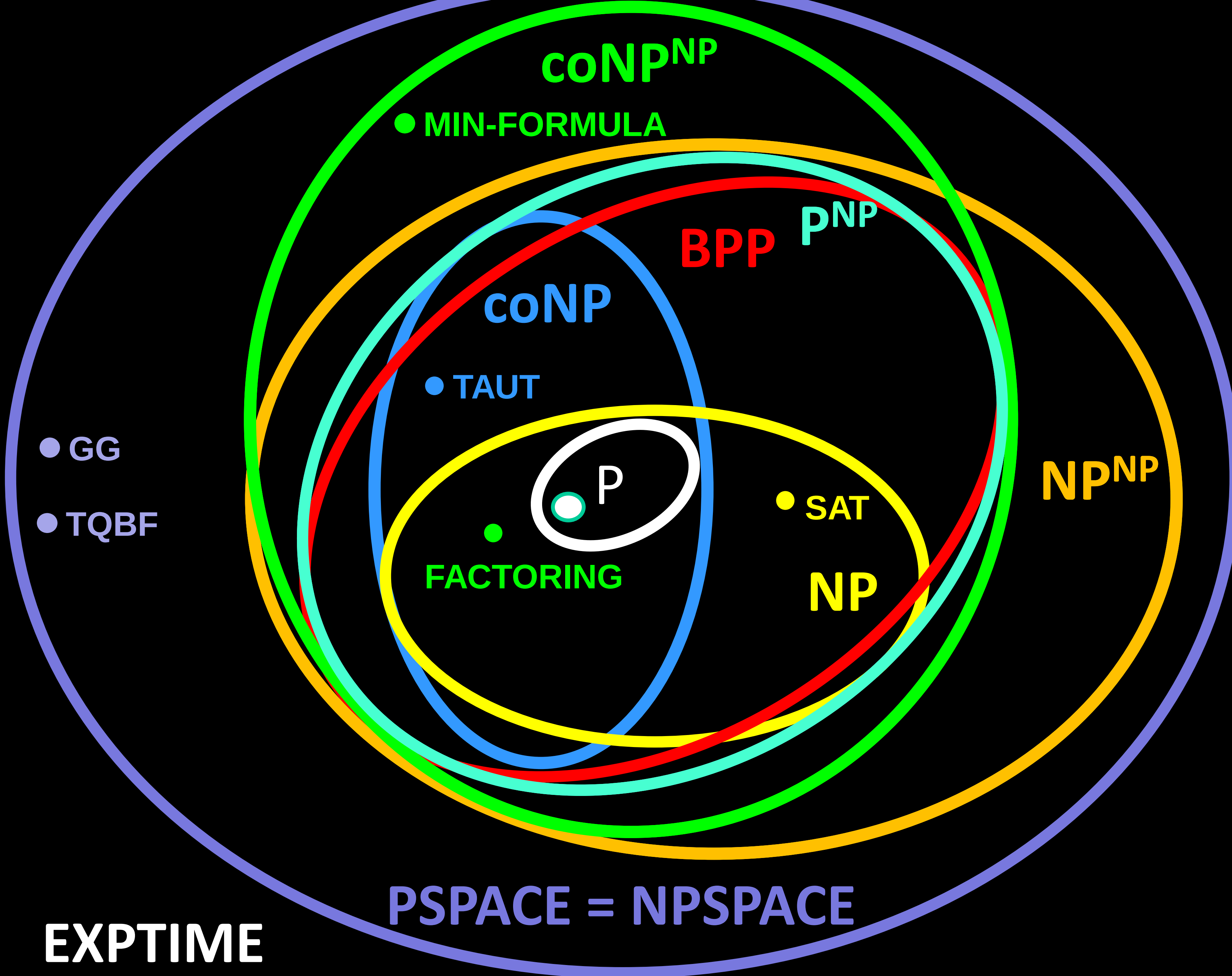
NP Completeness

coNP

Oracles:
P^{NP}, NP^{NP}, $coNP^{NP}$



EXPTIME



coNP^NP
MIN-FORMULA
P^NP
BPP
coNP
TAUT
GG
TQBF
P
SAT
NP^NP
FACTORING
NP
PSPACE = NPSPACE
EXPTIME

What's next?

A few possibilities…

6.046 – Design and Analysis of Algorithms

6.841/18.405 – Advanced Complexity Theory

18.408 – Topics in Theoretical Computer Science

18.416 – Randomized Algorithms

6.875 – Cryptography and Cryptanalysis

Many more! There's a big theory group at MIT!

Time to let the credits roll…

You have been watching:

6.045

Filmed at the

MASSACHVSETTS INSTITVTE OF TECHNOLOGY

in front of

absolutely nobody

Starring:

Ryan Williams

as “the professor”

A Large Hand Sanitizer Station

as itself