CS 154

Cook-Levin Theorem,
NP-Complete Problems
Is SAT solvable in O(n) time on a multitape TM?
Logic circuits of 6n gates for SAT?

If yes, then not only is P=NP, but there would be a “dream machine” that could crank out short proofs of theorems, quickly optimize all aspects of life... recognizing quality work is all you need to produce... THIS IS AN OPEN QUESTION!
Polynomial Time Reducibility

$f : \Sigma^* \rightarrow \Sigma^*$ is a polynomial time computable function
if there is a poly-time Turing machine $M$ that on every input $w$, halts with just $f(w)$ on its tape.

Language $A$ is poly-time reducible to language $B$, written as $A \leq_p B$,
if there is a poly-time computable $f : \Sigma^* \rightarrow \Sigma^*$ so that:

$$w \in A \iff f(w) \in B$$

$f$ is a polynomial time reduction from $A$ to $B$

Note there is a $k$ such that for all $w$, $|f(w)| \leq |w|^k$
f converts any string w into a string f(w) such that

\[ w \in A \iff f(w) \in B \]
Theorem: If $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$
Theorem: If $A \leq_p B$ and $B \in P$, then $A \in P$

Proof: Let $M_B$ be a poly-time TM that decides $B$. Let $f$ be a poly-time reduction from $A$ to $B$.

We build a machine $M_A$ that decides $A$ as follows:

$M_A = \text{On input } w,$

1. Compute $f(w)$
2. Run $M_B$ on $f(w)$, output its answer

$w \in A \iff f(w) \in B$
Theorem: If $A \leq_p B$ and $B \in \text{NP}$, then $A \in \text{NP}$

Proof: Analogous...
Theorem: If $A \leq_p B$ and $B \in P$, then $A \in P$

Theorem: If $A \leq_p B$ and $B \in NP$, then $A \in NP$

Corollary: If $A \leq_p B$ and $A \notin P$, then $B \notin P$
**Definition:** A language $B$ is **NP-complete** if:

1. $B \in \text{NP}$

2. Every $A$ in NP is poly-time reducible to $B$
   That is, $A \leq_p B$

When this is true, we say “$B$ is NP-hard”

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On homework, you showed
A language $L$ is recognizable iff $L \leq_m A_{\text{TM}}$

$A_{\text{TM}}$ is “**complete for recognizable languages**”:
$A_{\text{TM}}$ is recognizable, and for all recognizable $L$, $L \leq_m A_{\text{TM}}$
Suppose L is NP-Complete...

If \( L \in P \), then \( P = NP! \)

If \( L \notin P \), then \( P \neq NP! \)
Suppose $L$ is NP-Complete...

Then assuming the conjecture $P \neq NP$,

$L$ is not decidable in $n^k$ time, for every $k$
There are thousands of NP-complete problems!

Your favorite topic certainly has an NP-complete problem somewhere in it

Even the other sciences are not safe: biology, chemistry, physics have NP-complete problems too!
The Cook-Levin Theorem:
SAT and 3SAT are NP-complete

1. \(3\text{SAT} \in \text{NP}\)
   A satisfying assignment is a “proof” that a 3cnf formula is satisfiable

2. 3SAT is NP-hard
   Every language in NP can be polynomial-time reduced to 3SAT (complex logical formula)

Corollary: 3SAT \(\in\) P if and only if P = NP
Theorem (Cook-Levin): 3SAT is NP-complete

Proof Idea:

(1) 3SAT $\in$ NP (done)

(2) Every language $A$ in NP is polynomial time reducible to 3SAT (this is the challenge)

We give a poly-time reduction from $A$ to SAT

The reduction converts a string $w$ into a 3cnf formula $\phi$ such that $w \in A$ iff $\phi \in 3SAT$

For any $A \in NP$, let $N$ be a nondeterministic TM deciding $A$ in $n^k$ time

$\phi$ will simulate $N$ on $w$
Deterministic Computation

Nondeterministic Computation

accept or reject

\( n^k \)

accept

reject

\( \exp(n^k) \)
Let $L(N) \in \text{NTIME}(n^k)$. A tableau for $N$ on $w$ is an $n^k \times n^k$ matrix whose rows are the configurations of some possible computation history of $N$ on $w$.

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Each “cell” contains a $\sigma \in Q \cup \Gamma \cup \{\#\}$.
A tableau is **accepting** if the last row of the tableau is an accepting configuration

\[ \text{N accepts } w \text{ if and only if} \]
\[ \text{there is an accepting tableau for } N \text{ on } w \]

**Given** \( w \), we’ll construct a 3cnf formula \( \phi \) with \( O(|w|^{2k}) \) clauses, describing logical constraints that any accepting tableau for \( N \) on \( w \) must satisfy

The 3cnf formula \( \phi \) will be satisfiable **if and only if**
\[ \text{there is an accepting tableau for } N \text{ on } w \]
Variables of formula $\phi$ will \textit{encode} a tableau

Let $C = Q \cup \Gamma \cup \{ \# \}$

Each cell of a tableau contains a symbol from $C$

$cell[i,j] = \text{symbol in the cell at row } i \text{ and column } j$
$= \text{the jth symbol in the ith configuration}$

For every $i$ and $j$ ($1 \leq i, j \leq n^k$) and for every $s \in C$ we make a Boolean variable $x_{i,j,s}$ in $\phi$

Total number of variables $= |C|n^{2k}$, which is $O(n^{2k})$

The $x_{i,j,s}$ variables represent the cells of a tableau

We will enforce the condition: \textit{for all } i, j, s,

$x_{i,j,s} = 1 \iff cell[i,j] = s$
Idea: Make $\phi$ so that every satisfying assignment to the variables $x_{i,j,s}$ corresponds to an accepting tableau for $N$ on $w$ (an assignment to all cell[i,j]’s of the tableau)

The formula $\phi$ will be the AND of four CNF formulas:

$$\phi = \phi_{cell} \land \phi_{start} \land \phi_{accept} \land \phi_{move}$$

$\phi_{cell}$: for all $i$, $j$, there is a unique $s \in C$ with $x_{i,j,s} = 1$

$\phi_{start}$: the first row of the table equals the start configuration of $N$ on $w$

$\phi_{accept}$: the last row of the table has an accept state

$\phi_{move}$: every row is a configuration that yields the configuration on the next row
$\phi_{\text{start}}$ : the first row of the table equals the start configuration of $N$ on $w$

\[
\phi_{\text{start}} = x_{1,1,\#} \land x_{1,2,q_0} \land x_{1,3,w_1} \land x_{1,4,w_2} \land \cdots \land x_{1,n+2,w_n} \land x_{1,n+3,\Box} \land \cdots \land x_{1,n^k-1,\Box} \land x_{1,n^k,\#}
\]

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$O(n^k)$ clauses
$\phi_{\text{accept}}$ : the last row of the table has an accept state

$\phi_{\text{accept}} = \bigvee \ x_{n^k, j, q_{\text{accept}}}^{k, j, q_{\text{accept}}}$

$1 \leq j \leq n^k$

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$\phi_{\text{accept}}$ : the last row of the table has an accept state

\[
\phi_{\text{accept}} = \bigvee_{1 \leq j \leq n^k} x_{n^k,j, q_{\text{accept}}}
\]

How can we convert $\phi_{\text{accept}}$ into a 3-cnf formula?

The clause $(a_1 \lor a_2 \lor ... \lor a_t)$ is equivalent to

\[
(a_1 \lor a_2 \lor z_1) \land (\neg z_1 \lor a_3 \lor z_2) \land ... \land (\neg z_{t-3} \lor a_{t-1} \lor a_t)
\]

where $z_i$ are new variables.

This produces $O(t)$ new 3cnf clauses.

$O(n^k)$ clauses
\( \phi_{\text{cell}} : \) for all \( i, j \), there is a unique \( s \in C \) with \( x_{i,j,s} = 1 \)

\[
\phi_{\text{cell}} = \bigwedge_{1 \leq i, j \leq n^k} \left[ \left( \bigvee_{s \in C} x_{i,j,s} \right) \land \left( \bigwedge_{s,t \in C, s \neq t} (\neg x_{i,j,s} \lor \neg x_{i,j,t}) \right) \right]
\]

for all \( i,j \) \hspace{2cm} \text{at least one} \hspace{2cm} \text{at most one} \hspace{2cm} x_{i,j,s} \text{ is set to 1} \hspace{2cm} x_{i,j,s} \text{ is set to 1}

**O**(\( n^{2k} \)) clauses
Key Question: If one row yields the next row, how many cells can be different between the two rows?

Answer: AT MOST THREE CELLS!

\[ \phi_{\text{move}} : \text{every row is a configuration that yields the configuration on the next row} \]
**φ_{move}**: every row is a configuration that yields the configuration on the next row

**Key Question**: If one row yields the next row, how many cells can be different between the two rows?

**Answer**: AT MOST THREE CELLS!

\[
\begin{array}{cccccccc}
\# & b & a & a & q_1 & b & c & b & \# \\
\# & b & a & q_2 & a & c & c & b & \#
\end{array}
\]
$\phi_{\text{move}}$: every row is a configuration that yields the configuration on the next row

**Idea:** check that every $2 \times 3$ “window” of cells is **legal** (consistent with the transition function of $N$)
If $\delta(q_1,a) = \{(q_1,b,R)\}$ and $\delta(q_1,b) = \{(q_2,c,L), (q_2,a,R)\}$ which of the following windows are legal?
Key Lemma:

**IF** Every window of the tableau is legal, and
The 1\textsuperscript{st} row is the start configuration of N on w

**THEN** for all \( i = 1, \ldots, n^k - 1 \), the \( i \)th row of the tableau is
a configuration which yields the (i+1)th row.

**Proof Sketch:** (Strong) induction on \( i \).
The 1\textsuperscript{st} row is a configuration. If it didn’t yield the 2\textsuperscript{nd} row, there’s a 2 x 3 “illegal” window on 1\textsuperscript{st} and 2\textsuperscript{nd} rows
Assume rows 1,\ldots,L are all configurations which yield the next row, and assume every window is legal.
If row L+1 did not yield row L+2, then there’s a 2 x 3 window along those two rows which is “illegal”
The \((i, j)\) window of a tableau is the tuple \((a_1, \ldots, a_6) \in \mathbb{C}^6\) such that

\[
\begin{array}{cccc}
\text{row } i & \text{col. } j & \text{col. } j+1 & \text{col. } j+2 \\
\hline
a_1 & a_2 & a_3 \\
\hline
a_4 & a_5 & a_6 \\
\end{array}
\]
$\phi_{\text{move}}$ : every row is a configuration that legally follows from the previous configuration

$$
\phi_{\text{move}} = \bigwedge (\text{the (i, j) window is legal})
$$

$$
1 \leq i \leq n^k - 1
$$

$$
1 \leq j \leq n^k - 2
$$

$(\text{the (i, j) window is legal}) =$

$$
\bigvee (x_{i,j,a_1} \land x_{i,j+1,a_2} \land x_{i,j+2,a_3} \land x_{i+1,j,a_4} \land x_{i+1,j+1,a_5} \land x_{i+1,j+2,a_6})
$$

$(a_1, \ldots, a_6)$ is a legal window

$$
\equiv \bigwedge (\overline{x}_{i,j,a_1} \lor \overline{x}_{i,j+1,a_2} \lor \overline{x}_{i,j+2,a_3} \lor \overline{x}_{i+1,j,a_4} \lor \overline{x}_{i+1,j+1,a_5} \lor \overline{x}_{i+1,j+2,a_6})
$$

$(a_1, \ldots, a_6)$ is NOT a legal window
$\phi_{\text{move}} = \bigwedge (\text{the (i, j) window is "legal"})$

$1 \leq i, j \leq n^k$

the (i, j) window is “legal” =

$\equiv \bigwedge (a_1, \ldots, a_6) ISN'T "legal"

O(n^{2k})$ clauses
Summary. We wanted to prove:
Every A in NP has a polynomial time reduction to 3SAT

For every A in NP, we know A is decided by some nondeterministic $n^k$ time Turing machine N.

We gave a generic method to reduce a string $w$ to a 3CNF formula $\phi$ of $O(|w|^{2k})$ clauses such that satisfying assignments to the variables of $\phi$ directly correspond to accepting computation histories of N on $w$.

The formula $\phi$ is the AND of four 3CNF formulas:
$\phi = \phi_{\text{cell}} \land \phi_{\text{start}} \land \phi_{\text{accept}} \land \phi_{\text{move}}$
Theorem (Cook-Levin): SAT and 3SAT are NP-complete

Corollary: SAT ∈ P if and only if P = NP

Given a favorite problem Π ∈ NP, how can we prove it is NP-hard?

Generic Recipe:
1. Take a problem Σ that you know to be NP-hard (3-SAT)
2. Prove that Σ ≤ₚ Π

Then for all A ∈ NP, A ≤ₚ Σ and Σ ≤ₚ Π
We conclude that A ≤ₚ Π, and Π is NP-hard
\( \Pi \) is NP-Complete
The Clique Problem

Given a graph $G$ and positive $k$, does $G$ contain a complete subgraph on $k$ nodes?

$\text{CLIQUE} = \{ (G,k) \mid G \text{ is an undirected graph with a } k\text{-clique} \}$

Theorem (Karp): $\text{CLIQUE}$ is NP-complete
Proof Idea: $3\text{SAT} \leq_p \text{CLIQUE}$

Transform a 3-cnf formula $\phi$ into $(G,k)$ such that

$$\phi \in 3\text{SAT} \iff (G,k) \in \text{CLIQUE}$$

Want transformation that can be done in time that is \textit{polynomial in the length of $\phi$}

How can we encode a \textit{logic} problem as a \textit{graph} problem?
3SAT ≤ₚ CLIQUE

We transform a 3-cnf formula \( \phi \) into \((G,k)\) such that

\[ \phi \in 3SAT \iff (G,k) \in CLIQUE \]

Let \( C_1, C_2, \ldots, C_m \) be clauses of \( \phi \). Assign \( k := m \).

Make a graph \( G \) with \( m \) groups of 3 nodes each.

Group \( i \) corresponds to clause \( C_i \) of \( \phi \)

Each node in group \( i \) is labeled with a literal of \( C_i \)

Put edges between all pairs of nodes in different groups, except pairs of nodes with labels \( x_i \) and \( \neg x_i \)

Put no edges between nodes in the same group

When done putting in all the edges, erase the labels
\[(x_1 \lor x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2 \lor \neg x_2) \land (\neg x_1 \lor x_2 \lor x_2)\]
\[(x_1 \lor x_1 \lor x_1) \land (\neg x_1 \lor \neg x_1 \lor x_2) \land (x_2 \lor x_2 \lor x_2) \land (\neg x_2 \lor \neg x_2 \lor x_1)\]
Claim: $\phi \in 3\text{SAT} \Leftrightarrow (G, m) \in \text{CLIQUE}$

Claim: If $\phi \in 3\text{SAT}$ then $(G, m) \in \text{CLIQUE}$

Proof: Given a SAT assignment $A$ of $\phi$, for every clause $C$ there is at least one literal in $C$ that’s set true by $A$.
For each clause $C$, let $v_C$ be a vertex from group $C$ whose label is a literal that is set true by $A$.

Claim: $S = \{v_C : C \in \phi\}$ is an $m$-clique

Proof: Let $v_C, v_{C'}$ be in $S$. Suppose $(v_C, v_{C'}) \notin E$.
Then $v_C$ and $v_{C'}$ must label inconsistent literals, call them $x$ and $\neg x$.
But assignment $A$ cannot satisfy both $x$ and $\neg x$.
Therefore $(v_C, v_{C'}) \in E$, for all $v_C, v_{C'} \in S$.
Hence $S$ is an $m$-clique, and $(G, m) \in \text{CLIQUE}$
Claim: $\phi \in 3SAT \iff (G, m) \in CLIQUE$

Claim: If $(G, m) \in CLIQUE$ then $\phi \in 3SAT$

Proof: Let $S$ be an $m$-clique of $G$.

We construct a satisfying assignment $A$ of $\phi$.

Claim: $S$ contains exactly one node from each group.

Now for each variable $x$ of $\phi$, make assignment $A$:

Assign $x$ to 1 $\iff$ There is a vertex $v \in S$ with label $x$

For all $i = 1, \ldots, m$, at least one vertex from group $i$ is in $S$.

Therefore, for all $i = 1, \ldots, m$

$A$ satisfies at least one literal in the $i$th clause of $\phi$

Therefore $A$ is a satisfying assignment to $\phi$
Independent Set

**IS**: Given a graph $G = (V, E)$ and integer $k$, is there $S \subseteq V$ such that $|S| \geq k$ and

*no* two vertices in $S$ have an edge?

$IS = \{(G,k) \mid$ 

**CLIQUE**: Given $G = (V, E)$ and integer $k$, is there $S \subseteq V$ such that $|S| \geq k$

and every pair of vertices in $S$ have an edge?

**CLIQUE $\leq_p$ IS**: Given $G = (V, E)$, output $G' = (V, E')$ where

$E' = \{(u,v) \mid (u,v) \notin E\}$.

$(G, k) \in$ CLIQUE iff $(G', k) \in$ IS
Reading Assignment

Read Luca Trevisan’s notes for an alternative proof of the Cook-Levin Theorem!

Sketch:
1. Define **CIRCUIT-SAT**: Given a logical circuit $C(y)$, is there an input $a$ such that $C(a)=1$?
2. Show that **CIRCUIT-SAT** is NP-hard:
The $n^k \times n^k$ tableau for $N$ on $w$ can be simulated using a logical circuit of $O(n^{2k})$ gates
3. Reduce CIRCUIT-SAT to 3SAT in polytime
4. Conclude 3SAT is also NP-hard