Matrix-Vector Multiplication in Sub-Quadratic Time
(Some Preprocessing Required)

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Introduction

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Main Result: If we allow $O(n^{2+\varepsilon})$ preprocessing, then matrix-vector multiplication over any finite semiring can be done in $O(n^2/(\varepsilon \log n)^2)$.

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- Coppersmith and Winograd (1990): $O(n^{2.376})$ operations

  Not yet practical

Focus: *Combinatorial* Matrix Multiplication Algorithms

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More Non-Subtractive Boolean Matrix Mult. Algorithms:

- Atkinson and Santoro: $O(n^3/\log^{3/2} n)$ on a $(\log n)$-word RAM
- Rytter and Basch-Khanna-Motwani: $O(n^3/\log^2 n)$ on a RAM
- Chan: Four Russians can be implemented on $O(n^3/\log^2 n)$ on a pointer machine

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- **Preprocess** an $n \times n$ matrix A over a finite semiring in $O(n^{2+\varepsilon})$
- **Such that** vector multiplications with A can be done in $O(n^2/(\varepsilon \log n)^2)$

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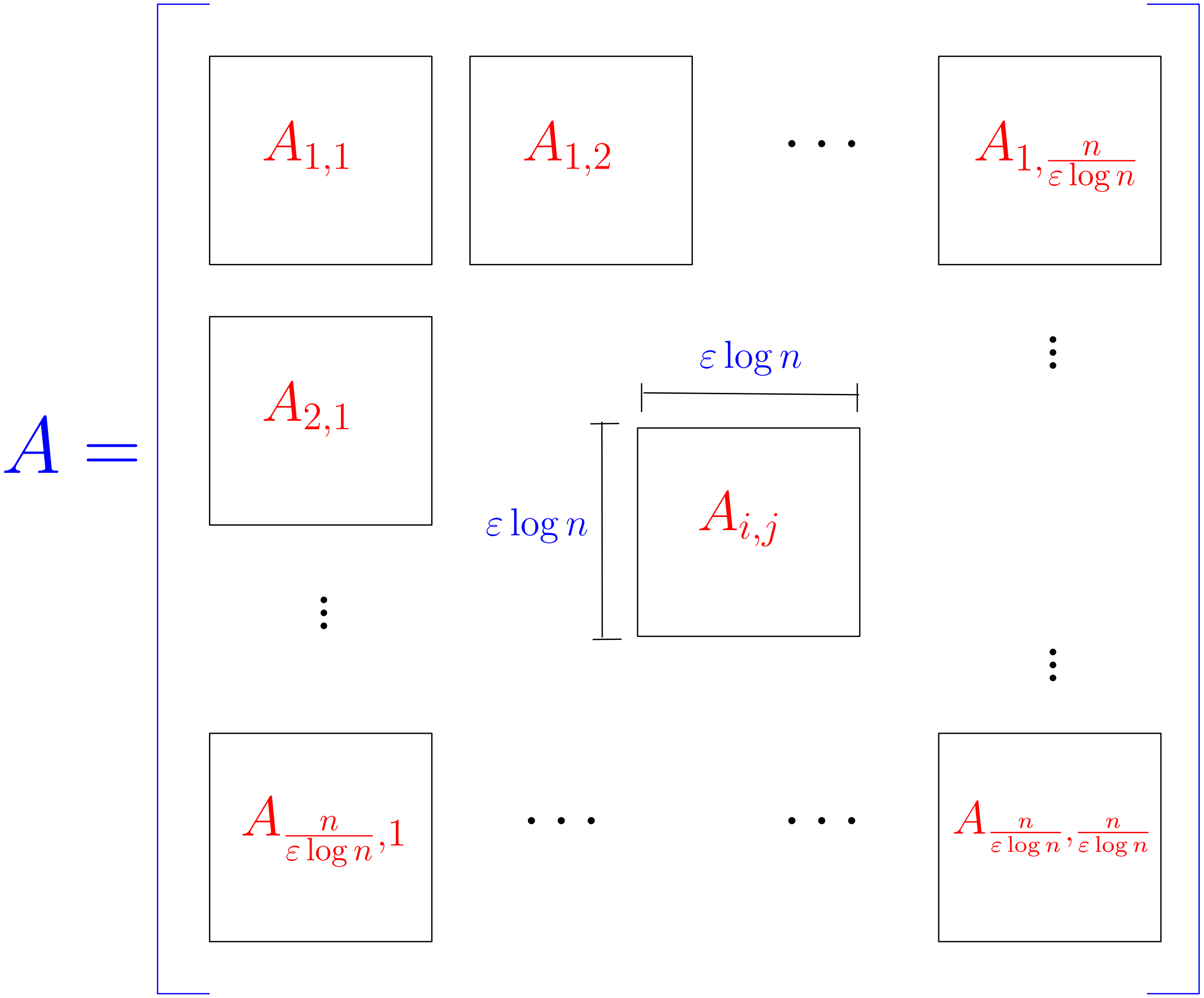
This Talk: The Boolean case

Preprocessing Phase: The Boolean Case

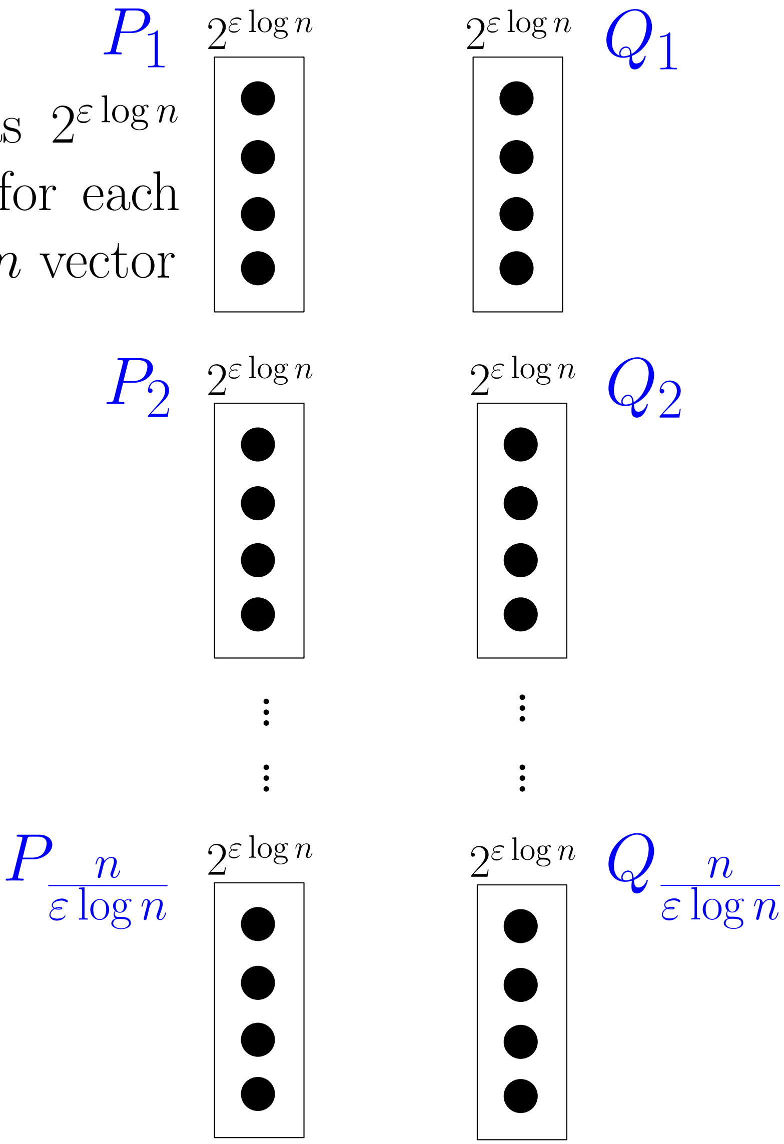
Partition the input matrix A into blocks of $\lceil \varepsilon \log n \rceil \times \lceil \varepsilon \log n \rceil$ size:



Preprocessing Phase: The Boolean Case

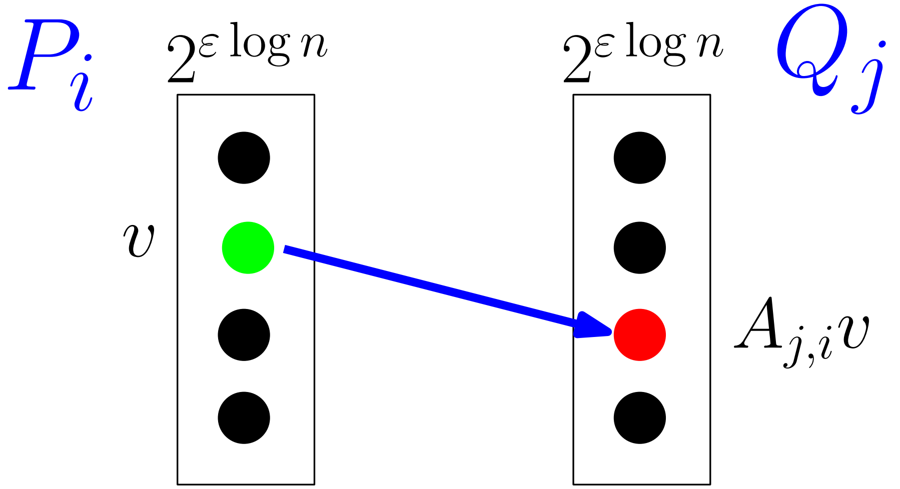
Build a graph G with parts $P_1, \ldots, P_{n/(\varepsilon \log n)}, Q_1, \ldots, Q_{n/(\varepsilon \log n)}$

Each part has $2^{\varepsilon \log n}$ vertices, one for each possible $\varepsilon \log n$ vector



Preprocessing Phase: The Boolean Case

Edges of G: Each vertex v in each P_i has exactly one edge into each Q_j



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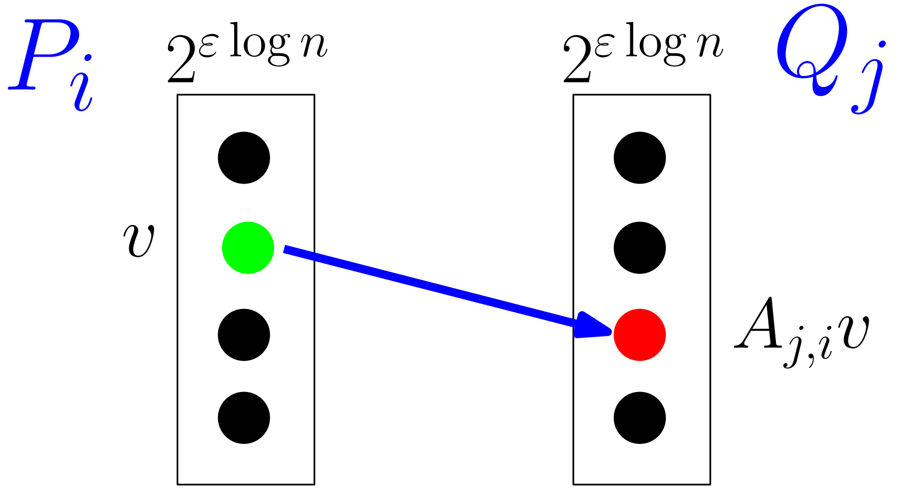

Time to build the graph:

$$\underbrace{\frac{n}{\varepsilon \log n}}_{\text{number of } Q_j} \cdot \underbrace{\frac{n}{\varepsilon \log n}}_{\text{number of } P_i} \cdot \underbrace{2^{\varepsilon \log n}}_{\text{number of nodes in } P_i} \cdot \underbrace{(\varepsilon \log n)^2}_{\text{matrix-vector mult of } A_{j,i} \text{ and } v} = O(n^{2+\varepsilon})$$

How to Do Fast Vector Multiplications

Let v be a column vector. Want: $A \cdot v$.

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(1) Break up v into $\varepsilon \log n$ sized chunks:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{\frac{n}{\varepsilon \log n}} \end{bmatrix}$$

How to Do Fast Vector Multiplications

(2) For each $i = 1, \ldots, n/(\varepsilon \log n)$, look up v_i in P_i.

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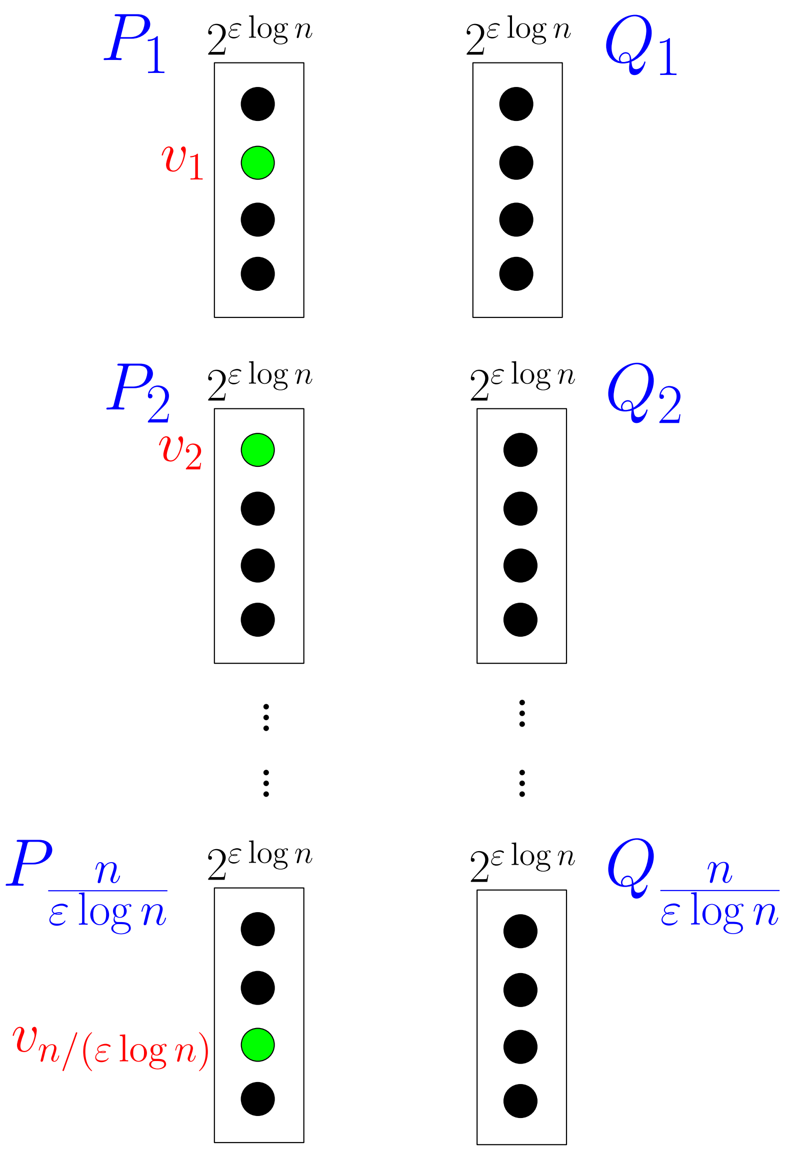

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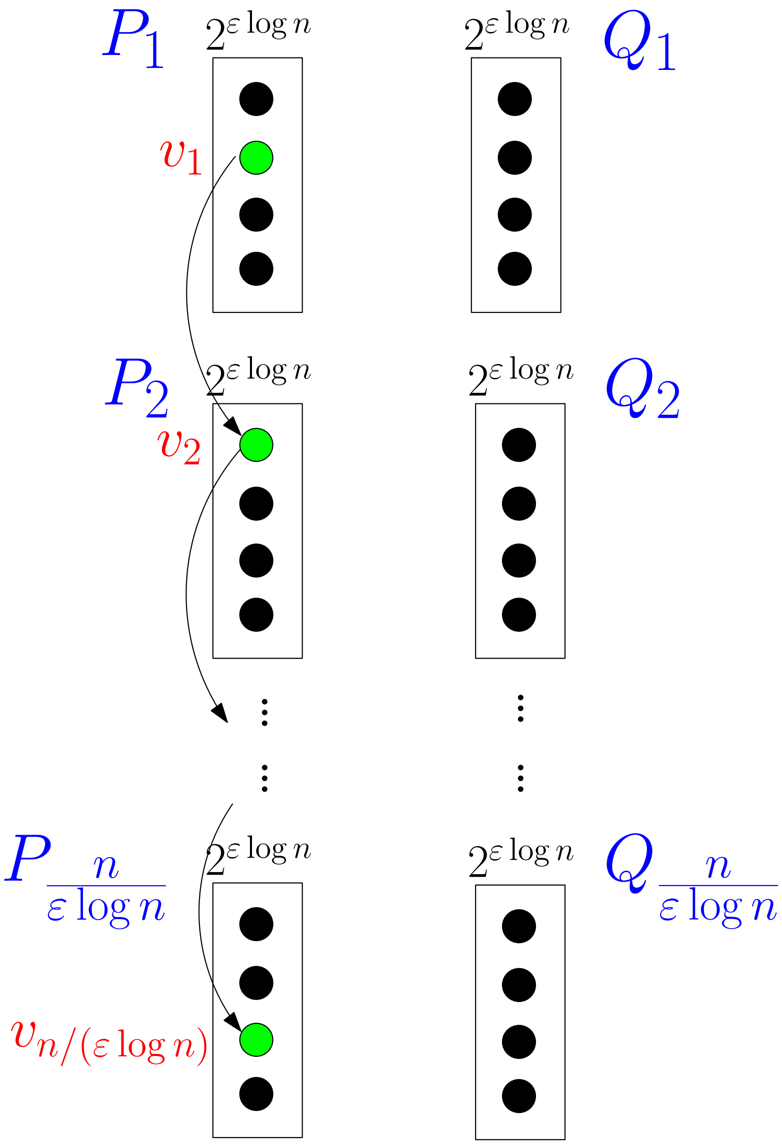

Takes $\tilde{O}(n)$ time.

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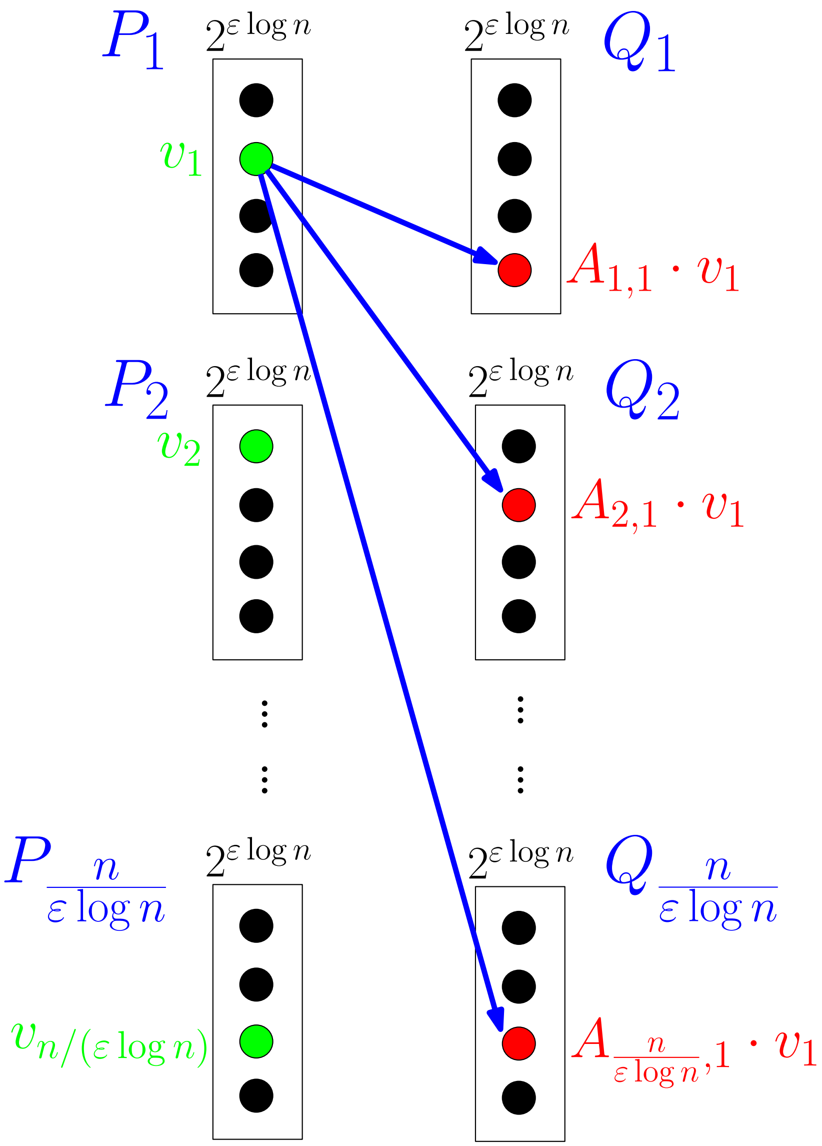

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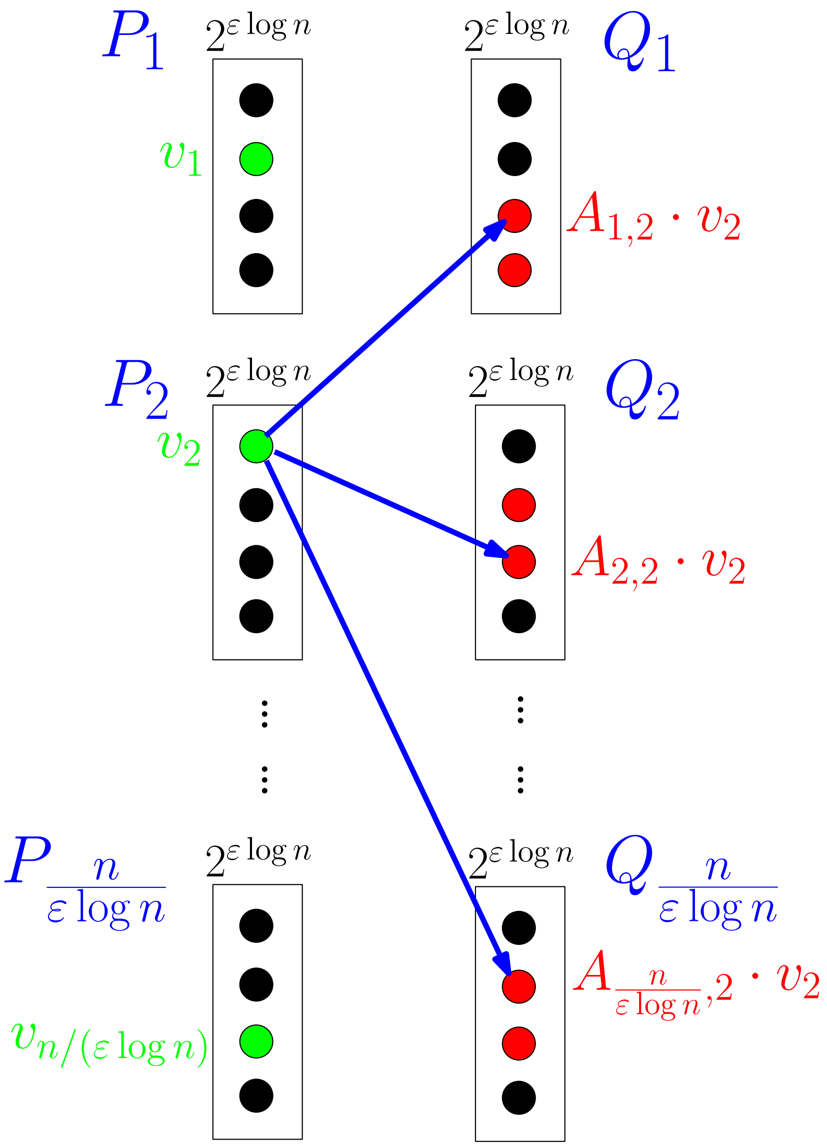

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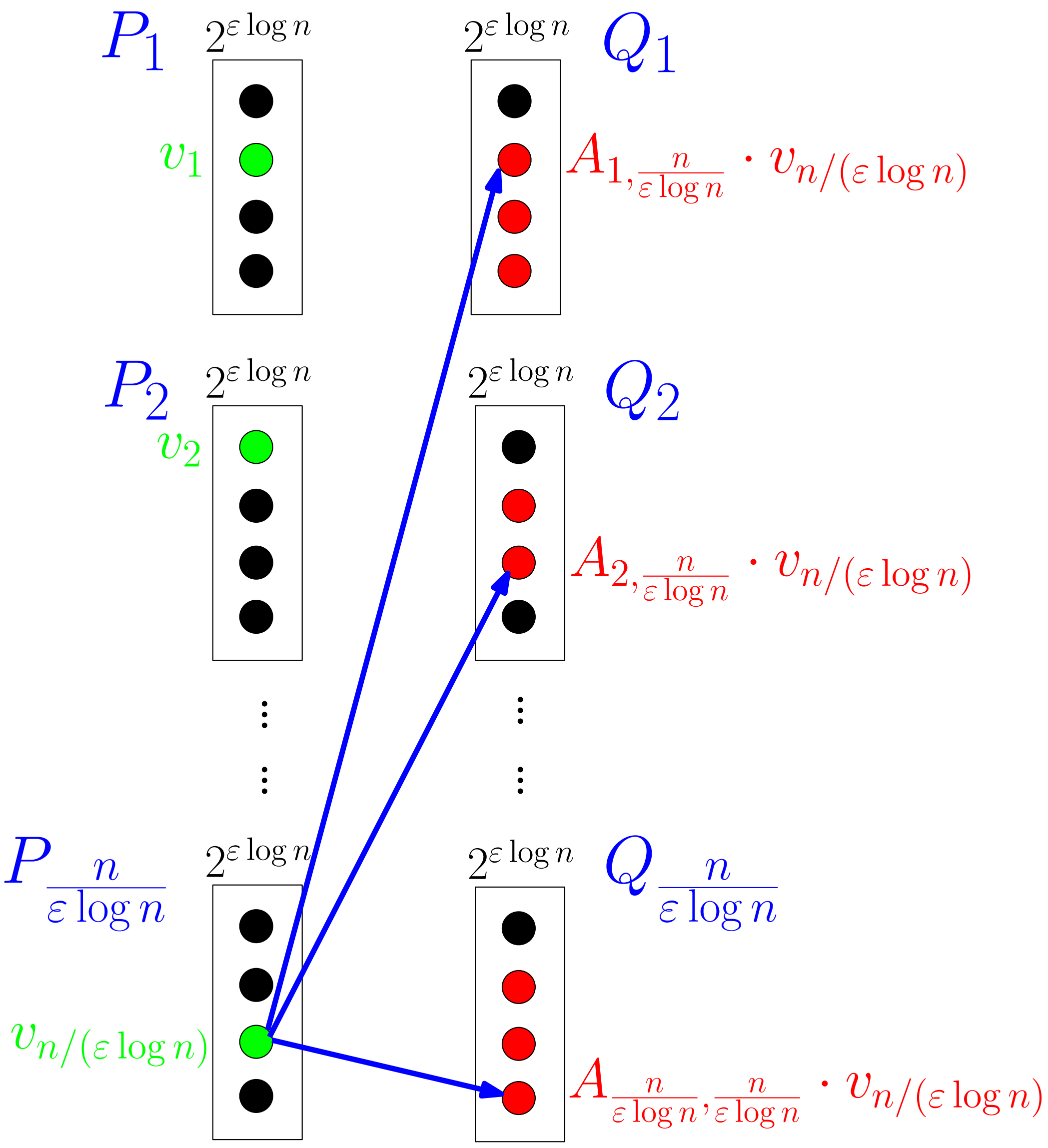
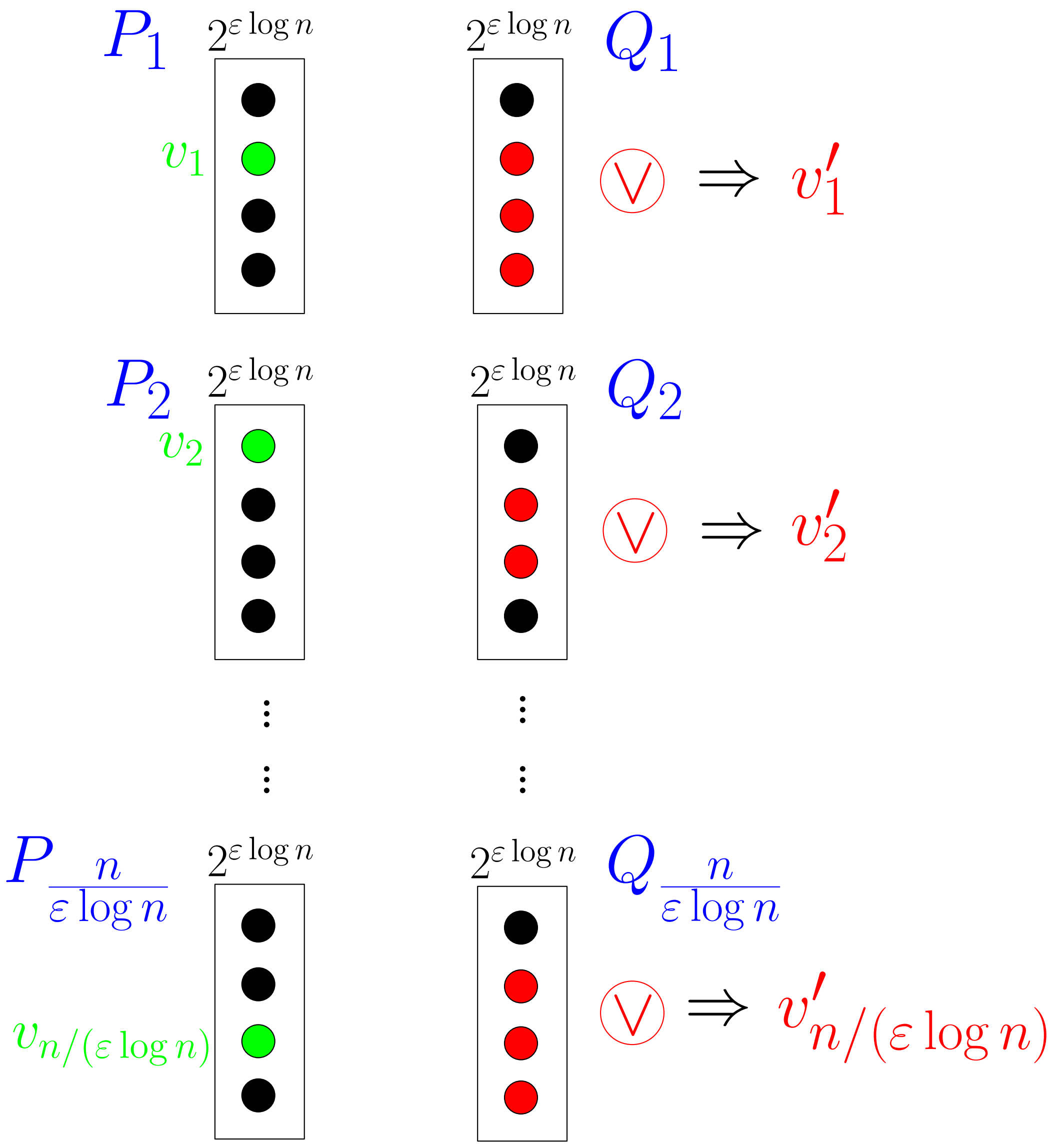


Takes $O\left(\left(\frac{n}{\varepsilon \log n}\right)^2\right)$

How to Do Fast Vector Multiplications

(4) For each Q_j, define v'_j as the **OR** of all marked vectors in Q_j



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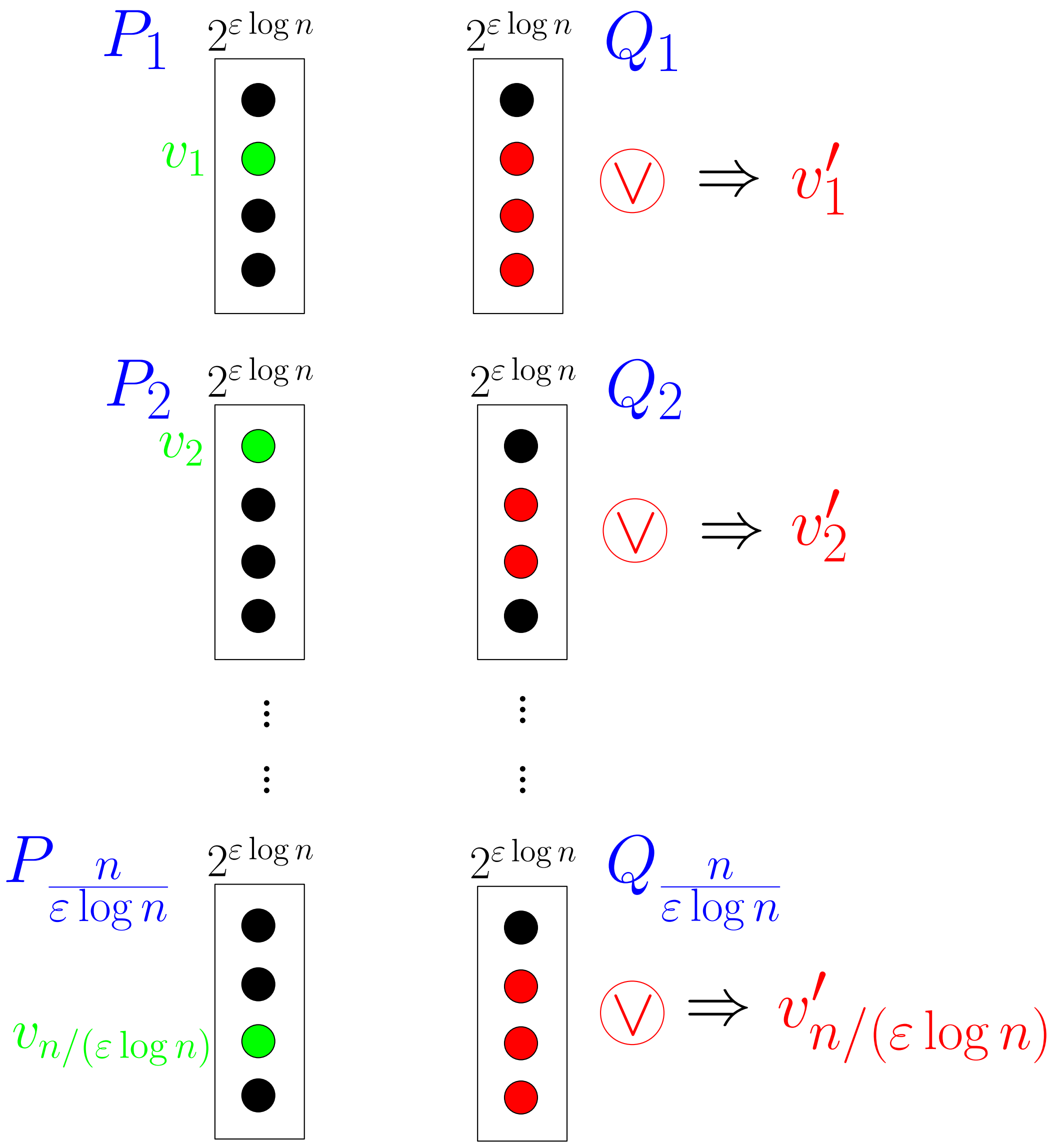

Takes $\tilde{O}(n^{1+\varepsilon})$ time

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$$Av = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n/(\varepsilon \log n)} \\ \vdots & \ddots & \vdots \\ A_{n/(\varepsilon \log n),1} & \cdots & A_{n/(\varepsilon \log n),n/(\varepsilon \log n)} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{\frac{n}{\varepsilon \log n}} \end{bmatrix}$$

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$$= \left(\bigvee_{i=1}^{n/(\varepsilon \log n)} A_{1,i} \cdot v_i, \ \ldots, \bigvee_{i=1}^{n/(\varepsilon \log n)} A_{1,n/(\varepsilon \log n)} \cdot v_i\right) = v'.$$

Some Applications

Can quickly compute the neighbors of arbitrary vertex subsets

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Proposition: $A \cdot v_S$ is the indicator vector for $N(S)$, the neighborhood of S.

Corollary: After $O(n^{2+\varepsilon})$ preprocessing, can determine the neighborhood of any vertex subset in $O(n^2/(\varepsilon \log n)^2)$ time.

(One level of BFS in $o(n^2)$ time)

Graph Queries

Corollary: After $O(n^{2+\varepsilon})$ preprocessing, can determine if a given vertex subset is an independent set, a vertex cover, or a dominating set, all in $O(n^2/(\varepsilon \log n)^2)$ time.

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Each can be quickly determined from knowing S and $N(S)$.

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Proof: Given vertex i, let S be its set of neighbors (gotten in $O(n)$ time).

S is *not* independent $\iff$ i participates in a triangle.

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- Can the algebraic matrix multiplication algorithms (Strassen, etc.) be applied to this problem?
- Can our ideas be extended to achieve non-subtractive Boolean matrix multiplication in $o(n^3/\log^2 n)$?

Thank you!