1 Supplementary Derivation of Reduced Poisson Formulation

Below, we show that the least square problem from Equation 5 in the paper can be reduced to the two matrix multiplications given in Equation 6.

$$\mathbf{t}(\mathbf{q}) = \underset{\mathbf{t}}{\operatorname{argmin}} \sum_{k\in 1\dots T} \sum_{j=2,3} \left\| \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j} - \mathbf{v}_{k,j} \right\|^2$$

where

$$\mathbf{v}_{k,j} = \begin{cases} \Phi_{k,j}\mathbf{t} - \Phi_{k,1}\mathbf{t} & \text{if } \mathbf{y}_{k,j} \notin F \text{ and } \mathbf{y}_{k,1} \notin F \\ \Phi_{k,j}\mathbf{t} - \Psi_{k,1}\mathbf{q} & \text{if only } \mathbf{y}_{k,1} \in F \\ \Psi_{k,j}\mathbf{q} - \Phi_{k,1}\mathbf{t} & \text{if only } \mathbf{y}_{k,j} \in F \end{cases}$$

Let $(k,j) \in F_0$ be the set of edges where neither $y_{k,j}$ nor $y_{k,1}$ are fixed.

Let $(k,j) \in F_1$ be the set of edges where only $y_{k,1}$ is fixed.

Let $(k,j) \in F_2$ be the set of edges where only $y_{k,j}$ is fixed. Then,

$$\begin{aligned} \mathbf{t}(\mathbf{q}) = \underset{\mathbf{t}}{\operatorname{argmin}} & \sum_{(k,j)\in F_0} \left\| \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j} - (\Phi_{k,j}\mathbf{t} - \Phi_{k,1}\mathbf{t}) \right\|^2 \\ & + \sum_{(k,j)\in F_1} \left\| \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j} - (\Phi_{k,j}\mathbf{t} - \Psi_{k,1}\mathbf{q}) \right\|^2 \\ & + \sum_{(k,j)\in F_2} \left\| \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j} - (\Psi_{k,j}\mathbf{q} - \Phi_{k,1}\mathbf{t}) \right\|^2 \end{aligned}$$

Taking the first order conditions, we obtain,

$$\begin{aligned} & \sum_{(k,j)\in F_0} (\Phi_{k,j} - \Phi_{k,1})^T((\Phi_{k,j} - \Phi_{k,1})\mathbf{t} - \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}) \\ + & \sum_{(k,j)\in F_1} {\Phi_{k,j}}^T(\Phi_{k,j}\mathbf{t} - \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j} + \Psi_{k,1}\mathbf{q}) \\ + & \sum_{(k,j)\in F_2} {\Phi_{k,1}}^T(\Phi_{k,1}\mathbf{t} + \sum_{\ell\in 1\dots P} \beta_{k,\ell} D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j} - \Psi_{k,j}\mathbf{q}) = 0. \end{aligned}$$

Grouping terms with $\mathbf{t}$ on the left hand side gives,

$$
\begin{aligned}
&\Big(\sum_{(k,j)\in F_0}(\Phi_{k,j}-\Phi_{k,1})^T(\Phi_{k,j}-\Phi_{k,1})+\sum_{(k,j)\in F_1}\Phi_{k,j}^T\Phi_{k,j}+\sum_{(k,j)\in F_2}\Phi_{k,1}^T\Phi_{k,1}\Big)\mathbf{t}\\
&=\sum_{(k,j)\in F_0}(\Phi_{k,j}-\Phi_{k,1})^T\sum_{\ell\in 1\ldots P}\beta_{k,\ell}D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}\\
&+\sum_{(k,j)\in F_1}{\Phi_{k,j}}^T\sum_{\ell\in 1\ldots P}\beta_{k,\ell}D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}\\
&+\sum_{(k,j)\in F_2}{\Phi_{k,1}}^T\sum_{\ell\in 1\ldots P}\beta_{k,\ell}D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}\\
&+\sum_{(k,j)\in F_1}{\Phi_{k,j}}^T\Psi_{k,1}\mathbf{q}+\sum_{(k,j)\in F_2}{\Phi_{k,1}}^T\Psi_{k,j}\mathbf{q}.
\end{aligned}
$$

The equation above takes the form

$$A\mathbf{t}=\mathbf{b}_1(D(\mathbf{q}))+B_2\mathbf{q}$$

where,

$$
\begin{aligned}
A\in\mathbb{R}^{12P\times 12P}&=\sum_{(k,j)\in F_0}(\Phi_{k,j}-\Phi_{k,1})^T(\Phi_{k,j}-\Phi_{k,1})\\
&+\sum_{(k,j)\in F_1}\Phi_{k,j}^T\Phi_{k,j}+\sum_{(k,j)\in F_2}\Phi_{k,1}^T\Phi_{k,1}\\
B_2\in\mathbb{R}^{12P\times 12J}&=\sum_{(k,j)\in F_1}(\Phi_{k,j})^T\Psi_{k,1}+\sum_{(k,j)\in F_2}(\Phi_{k,1})^T\Psi_{k,j}
\end{aligned}
$$

The expression $\mathbf{b}_1(D(\mathbf{q}))$ can be simplified further:

$$
\begin{aligned}
\mathbf{b}_1(D(\mathbf{q}))&=\sum_{(k,j)\in F_0}(\Phi_{k,j}-\Phi_{k,1})^T\sum_{\ell\in 1\ldots P}\beta_{k,\ell}D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}\\
&+\sum_{(k,j)\in F_1}{\Phi_{k,j}}^T\sum_{\ell\in 1\ldots P}\beta_{k,\ell}D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}\\
&+\sum_{(k,j)\in F_2}{\Phi_{k,1}}^T\sum_{\ell\in 1\ldots P}\beta_{k,\ell}D_\ell(\mathbf{q})\hat{\mathbf{v}}_{k,j}
\end{aligned}
$$

Moving the sum over ℓ outside and applying the identity,

$$\mathbf{vec}(ABC)=(C^T\otimes A)\mathbf{vec}(B)\text{, gives,}$$

$$
\begin{aligned}
&= \sum_{\ell \in 1 \ldots P} \Big(\sum_{(k,j) \in F_0} (\Phi_{k,j} - \Phi_{k,1})^T \beta_{k,\ell} (\hat{\mathbf{v}}_{k,j}^T \otimes I_{3\times 3}) \\
&\qquad + \sum_{(k,j) \in F_1} \Phi_{k,j}{}^T \beta_{k,\ell} (\hat{\mathbf{v}}_{k,j}^T \otimes I_{3\times 3}) \\
&\qquad - \sum_{(k,j) \in F_2} \Phi_{k,1}{}^T \beta_{k,\ell} (\hat{\mathbf{v}}_{k,j}^T \otimes I_{3\times 3}) \Big) \mathbf{vec}(D_\ell(\mathbf{q}))
\end{aligned}
$$

where $\otimes$ denotes the Kronecker product.

$$
\begin{aligned}
&= B_1 [\mathbf{vec}(D_1(\mathbf{q}))^T \ldots \mathbf{vec}(D_P(\mathbf{q}))^T]^T \\
&= B_1 \mathbf{d}(\mathbf{q})
\end{aligned}
$$

where,

$$
\begin{aligned}
B_{1\ell} = &\sum_{(k,j) \in F_0} (\Phi_{k,j} - \Phi_{k,1})^T \beta_{k\ell} (\hat{\mathbf{v}}_{k,j}^T \otimes I_{3\times 3}) \\
&+ \sum_{(k,j) \in F_1} \Phi_{k,j}^T \beta_{k\ell} (\hat{\mathbf{v}}_{k,j}^T \otimes I_{3\times 3}) \\
&- \sum_{(k,j) \in F_2} \Phi_{k,1}^T \beta_{k\ell} (\hat{\mathbf{v}}_{k,j}^T \otimes I_{3\times 3}) \\
B_1 = &[B_{11} \ldots B_{1P}].
\end{aligned}
$$

We now have,

$$
A\mathbf{t} = B_1 \mathbf{d}(\mathbf{q}) + B_2 \mathbf{q}
$$

Solving for $\mathbf{t}$, we obtain,

$$
\begin{gathered}
C_1 \in \mathbb{R}^{12P \times 9P} = A^{-1} B_1; \qquad C_2 \in \mathbb{R}^{12P \times 12J} = A^{-1} B_2 \\
\mathbf{t}(\mathbf{q}) = C_1 \mathbf{d}(\mathbf{q}) + C_2 \mathbf{q}.
\end{gathered}
$$

The matrices C_1 and C_2 are constant and are precomputed. Thus, solving the least squares problem given in Equation 5 involves just two matrix multiplications.