Supplement to
"Dead-End Elimination as a Heuristic for Min-Cut Image Segmentation"

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October 2006

Abstract

This article is a supplement to our 2006 ICIP paper, "Dead-End Elimination as a Heuristic for Min-Cut Image Segmentation" [1], which we assume the reader has read. We summarize the proof of the dead-end elimination theorem (due to Desmet *et al.* [2] and Goldstein [3]), briefly discuss the performance of our current implementation of DEE pairs, show the input images for which timings are referenced in the main paper, and present examples of processed images to show that DEE does not affect the resulting segmentation.

1 Proof of DEE Theorem

The original dead-end elimination theorem is due to Desmet *et al.* [2]. Goldstein's DEE theorem [3] is closely related but more powerful: Let i_a and i_r be two specific assignments at a particular position i. Then, if

$$E(i_a) - E(i_b) + \sum_j \min_f [E(i_a, j_f) - E(i_b, j_f)] > 0,$$

the assignment i_a cannot possibly be in the global minimum configuration and can therefore be eliminated from the space. i_a cannot be in the global minimum energy assignment if there exists another assignment at the same position, i_b, such that the total energy with i_a is higher than the total energy with i_b even when we choose every other position to give i_a the best pairwise energies relative to i_b.

We now summarize proofs due to Desmet and Goldstein.

Proof. Given two possible assignments, i_a and i_b at position i, let us assume that

$$E(i_a) - E(i_b) + \sum_j \min_f [E(i_a, j_f) - E(i_b, j_f)] > 0$$

This is the premise of the DEE theorem. Let the global minimum energy assignment (GMEA) at each position be represented by the subscript g. Define

$$E_{tot,g}(i_a) = E(i_a) + \sum_j (i_a, j_g) + \sum_j E(j_g) + \sum_{j<k} E(j_g, k_g)$$
$$E_{tot,g}(i_b) = E(i_b) + \sum_j (i_b, j_g) + \sum_j E(j_g) + \sum_{j<k} E(j_g, k_g)$$
$$\text{where } j,k \neq i$$

Here, $E_{tot,g}(i_f)$ is the total energy of the system when every position except position i has the assignment it has in the GMEA, while the i^{th} position is assigned f. We pulled out terms that involved the i^{th} position in each of the above two definitions. Subtracting these two equations and canceling out the terms that do not involve the i^{th} position yields:

$$\begin{aligned} &E_{tot,g}(i_a) - E_{tot,g}(i_b) \\ &= E(i_a) - E(i_b) + \sum_j E(i_a, j_g) - \sum_j E(i_b, j_g) \\ &= E(i_a - E(i_b) + \sum_j [E(i_a, j_g) - E(i_b, j_g)]) \end{aligned}$$

Note that for each j,

$$E(i_a, j_g) - E(i_b, j_g) \geq \min_f [E(i_a, j_f) - E(i_b, j_f)],$$

by the definition of the minimum operator. Thus, the sum over all j can be bounded:

$$\sum_j [E(i_a, j_g) - E(i_b, j_g)] \geq \sum_j \min_f [E(i_a, j_f) - E(i_b, j_f)]$$

Therefore,

$$E_{tot,g}(i_a) - E_{tot,g}(i_b) \geq E(i_a) - E(i_b) + \sum_j \min_f [E(i_a, j_f) - E(i_b, j_f)] > 0$$

by our original assumption, which forms the premise of the DEE theorem. It follows that $E_{tot,g}(i_a) > E_{tot,g}(i_b)$. But this means that i_a cannot possibly be an assignment in the GMEA because there exists an assignment i_b that, when all other positions have GMEA assignments, produces a lower energy assignment. □

2 Input Images and Segmentation Results

We ran the Edmonds-Karp and Boykov-Kolmogorov min-cut algorithms on a set of natural images of size 50×50 pixels and 500×500 pixels. We show a subset of these in Figure 1; these are the input images for which timings are referenced in Table 1 of the main paper.

3 Performance of DEE Pairs

DEE pairs eliminates pairs of assignments, such that when DEE singles is run afterwards, there are fewer degrees of freedom in choosing assignments over neighbors, and more pixels can therefore be eliminated. The pseudocode and runtime of DEE pairs is given in the mai paper.

When used with DEE singles, our current implementation of DEE pairs did not offer consistent additional speed-ups over Edmonds-Karp (EK) [4] alone, and further slowed down Boykov-Kolmogorov (BK) [5]. However, we have found that the structure of the image segmentation problem will allow us to further speed up DEE pairs, and we are

Input	DEE singles+EK	DEE singles+pairs+EK	ΔEP
$bird1_{100}$	0.17	0.19	1.09%
$bird2_{100}$	0.49	0.10	23.29%
cat_{100}	0.83	0.72	1.36%
$coast_{100}$	0.11	0.11	1.35%
$feather_{100}$	0.26	0.23	1.70%
$fungus_{100}$	0.05	0.04	0.82%
$monkey_{100}$	0.05	0.07	0.65%
$rock1_{100}$	0.09	0.05	0.59%
$rock2_{100}$	0.13	0.13	0.94%
$rowboat_{100}$	0.05	0.05	0.21%
$squirrel_{100}$	0.14	0.13	0.56%
$windmill_{100}$	0.17	0.16	1.35%

Table 1: Comparison of running times (in seconds) for DEE+EK with and without DEE pairs. All images used here were 100x100 pixels. Times do not include time to read in initial image data. "ΔEP" is the (Percentage of pixels assigned using DEE pairs - Percentage assigned without DEE pairs)

currently doing so. For example, for this problem, the pairs of assignments that generally increase eliminating power when running DEE singles are those for which both pixels are assigned to the same segment. Therefore, by focusing on eliminating only these pairs, we can cut the runtime of DEE pairs in half.

Shown in Table 1 are running times for 100x100 pixel images for (a) DEE singles followed by EK, and (b) DEE singles followed by one iteration of DEE pairs, DEE singles again, and finally EK. In all cases DEE singles was run until no more pixels could be eliminated by it. Note that for some images (e.g. `bird2`, `cat`) DEE pairs sped up the segmentation, but the speedup was not consistent across all images. For most images, one iteration of DEE pairs eliminated enough pairs to ultimately assign only around 0-2% more pixels than DEE singles could alone.

References

[1] Mala L. Radhakrishnan and Sara L. Su, "Dead-end elimination as a heuristic for min-cut image segmentation," in *Proceedings of the 13th IEEE International Conference on Image Processing*, 2006.

[2] Johan Desmet, Marc De Maeyer, Bart Hazes, and Ignace Lasters, "The dead-end elimination theorem and its use in protein side-chain positioning," *Nature*, vol. 356, pp. 539–542, April 1992.

[3] Robert F. Goldstein, "Efficient rotamer elimination applied to protein side-chains and related spin glasses," *Biophysical Journal*, pp. 1135–1140, May 1994.

[4] Jack Edmonds and Richard M. Karp, "Theoretical improvements in algorithmic efficiency for network flow problems," *Journal of the ACM*, vol. 19, no. 2, pp. 248–264, April 1972.

[5] Yuri Boykov and Vladimir Kolmogorov, "An experimental comparison of min-cut/max-flow algorithms for energy minimization in computer vision," *IEEE Transactions on Pattern Analysis*, September 2004.

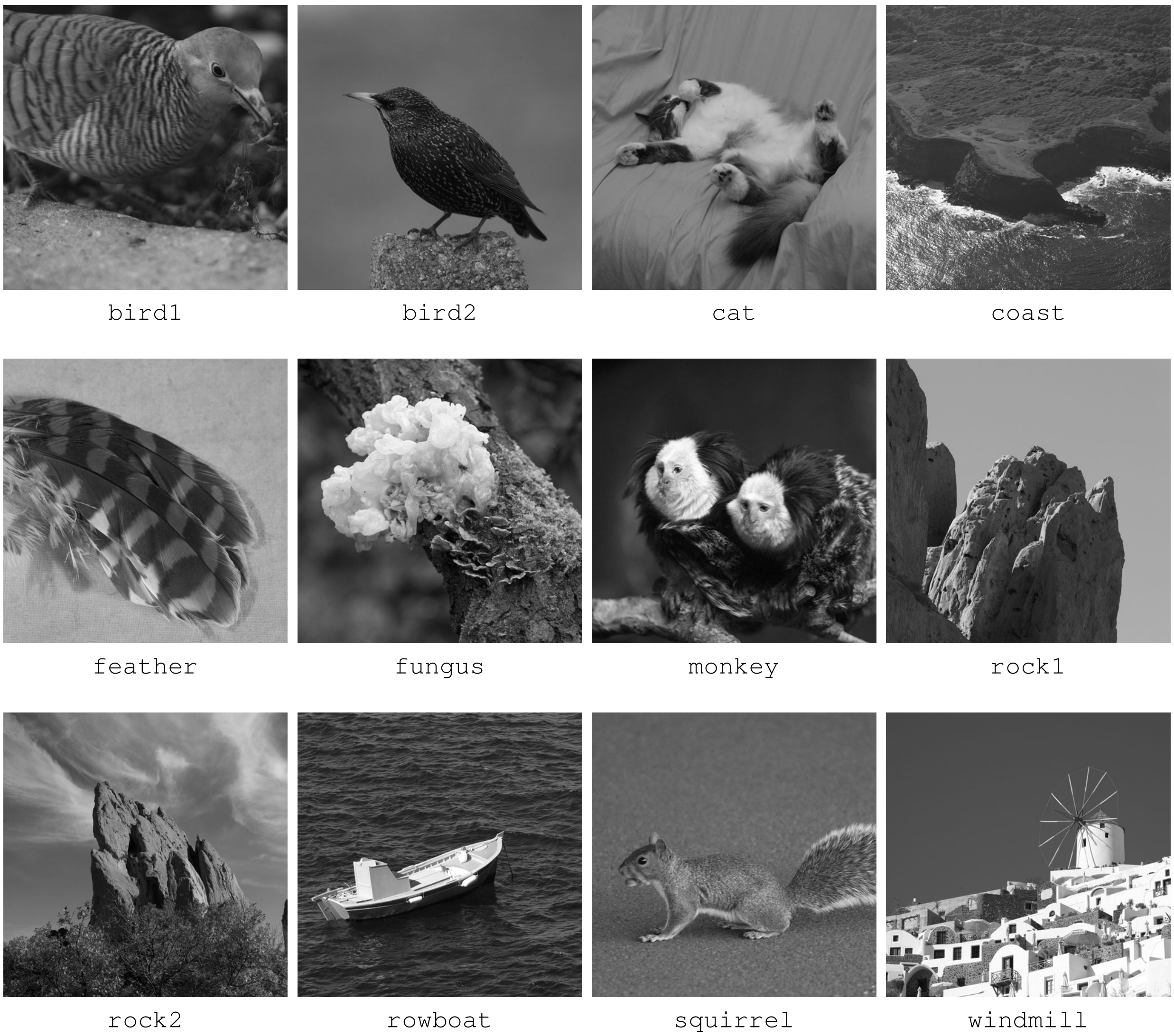
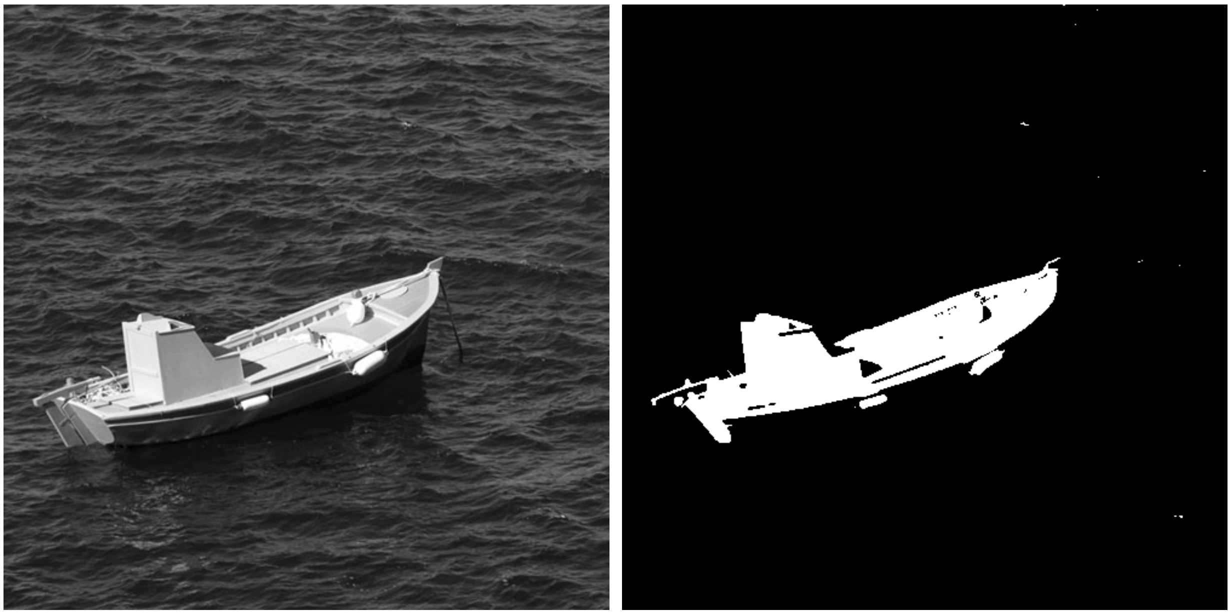
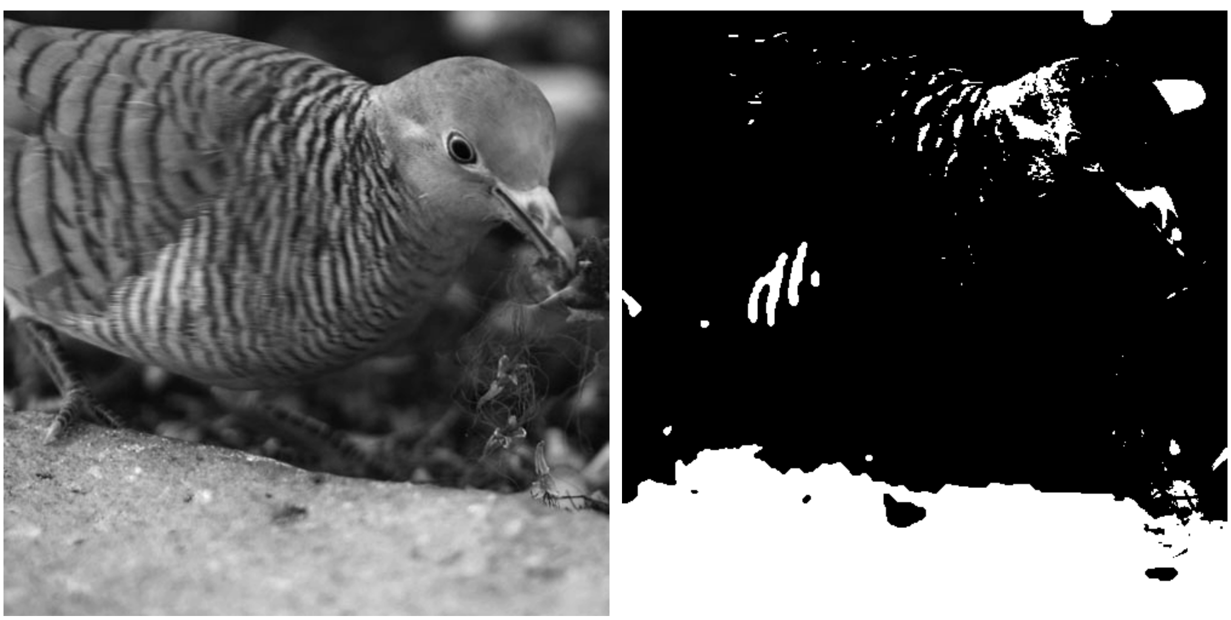


Figure 1: Input images.



(a) `rowboat`: Successful segmentation.



(a) `bird1`: Unsuccessful segmentation.

Figure 2: Successful and unsuccessful binary segmentations of 500×500 images. Both min-cut algorithms (EK and BK), alone and preceded by DEE, produce identical segmentation results.