Inflating the Cube by Shrinking

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Abstract

We present a continuous, submetric deformation of the surface of the cube which increases the enclosed volume by about 25.67%. This improves the previous bound of about 21.87% by Bleecker

1 Introduction

We address the problem how large the volume of a body with a surface isometric to that of the unit cube can be. The idea of considering volume-increasing isometric embeddings is due to Bleecker [2]. He proved that a volume-increasing continuous isometric deformation exists for every simplicial convex surface in $\mathbb{R}^3$. A deformation is called isometric if it preserves the geodesic distances between any two points on the surface. Bleecker also gives a direct construction for the cube and other platonic solids. By Alexandrov's uniqueness theorem [1] a body resulting from such a deformation must be non-convex.

Most recently, Pak [6] gave an easy construction for increasing the volume of the unit cube to about 1.1812 based on the work of Milka [4]. Bleecker's more involved construction yields a volume of about 1.2187. A simple upper bound can be obtained by the volume of the sphere which has the same surface as the cube. This gives an upper bound on the volume of 1.3820. This bound is not sharp as the cones around cube vertices are not isometric to spherical regions.

Bleecker conjectured that for every (not necessarily simplicial) polyhedron $P \subset \mathbb{R}^3$ there exists a volume-increasing deformation of ∂P [2]. Bleecker's conjecture was positively resolved by Pak [5], who also extended it to non-convex polyhedra and polyhedra in higher dimensions.

It was observed by Pak [6] that one can also consider submetric embeddings, a larger class containing the isometric embeddings. In a submetric embedding geodesic distances on the surface are non-increasing. By a result of Burago and Zalgaller [3], for every submetric embedding there is an isometric embedding arbitrary close to it. Thus the bound achieved by submetric embeddings coincides with that by isometric embeddings.

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In this paper, we present a shrinking, i.e., a continuous, submetric deformation of the unit cube for which the resulting volume is at least 1.2567. This also improves the lower bound on the volume of a surface isometric to that of a unit cube. The shrinking problem and the idea of looking at shrinkings in order to get isometric embeddings is due to Pak [5].

2 A Shrinking of the Cube with Large Volume

We present volume-increasing submetric embeddings of the cube. The embeddings are parametrized by $\varepsilon \in [0, 0.5]$. Increasing ε from zero yields a continuous deformation. In this section we show a construction with volume about 1.2444, which we improve in the next section further.

Our approach is a refinement of Igor Pak's work [5, 6]. The original cube is given as the convex hull of the set $\{0,1\}^3$. We denote vertices on the surface of the cube by p_i. The same vertex in the deformed cube is denoted as v_i.

As a first step we cut off ε-cubes in every corner of the cube (see Figure 1) Now we are going to deform

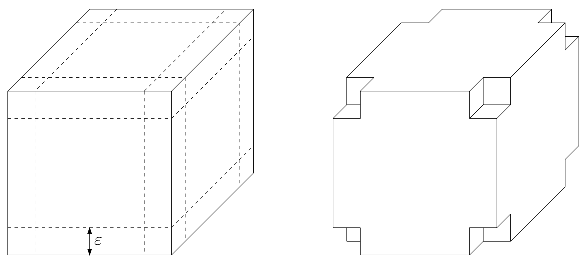

Figure 1: Cutting off ε cubes.

the remaining part of the cube. We place one vertex in the middle of every ε segment as shown in Figure 2.a. The segments defined between $p_1, p_{5/4}, p_{3/2}, p_{7/4}, p_2$ have the length $\varepsilon/2$. Let the framework induced by this chain be C. We move the vertices of C such that $v_1, \ldots, v_2$ lie on a quarter-circle (depicted in Figure 2.b). We apply the deformation for all corresponding pairs of ε segments. This leads to a body which we divide into a *corpus* and 12 *bars*. Figure 3 shows the parts. A bar is a prism with a 6-gon as base area. The 6-gon is inscribed in quarter circle. Its shorter edges have the length $\varepsilon/2$. The radius of the quarter circle is denoted by δ. Expressed in terms of ε we

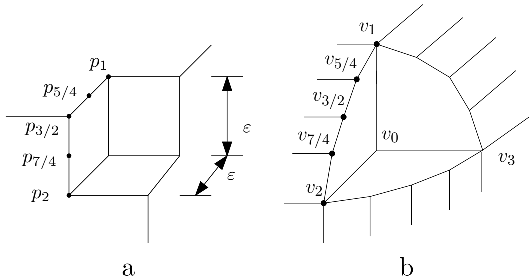

Figure 2: Bending the chains induced by a pair of ε segments.

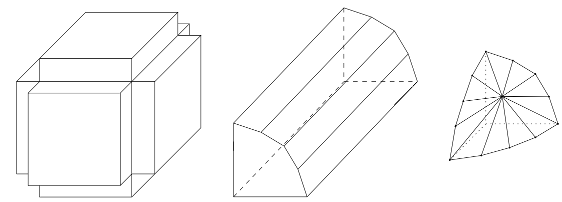

Figure 3: Corpus and bar and star.

obtain

$$\delta = \frac{1}{2}\varepsilon\sqrt{\frac{1}{2-2\cos(\pi/8)}}.$$

The volume of one bar is given by

$$V_{bar} = 2(1-2\varepsilon)\delta^2\sin(\pi/8).$$

The corpus is the remaining part after cutting out the bars. Its volume is given by

$$V_{corpus} = (1-2\varepsilon)^3 \; + \; 6\delta(1-2\varepsilon)^2.$$

It remains to place the cut-outs at the corners of the body. We have to deform the ε-cubes, such that they fit into the open 12-gons (formed by three chains C) of the body. Consider the open part depicted in Figure 2.b. One vertex is part of the corpus and the three bars, which we denote by v_0. In the following we refer to an orthogonal coordinate system. Its origin lies at v_0 and its x,y, and z directions are defined by the rays passing through v_1, v_2, and v_3. The object we glue into this part is called *star*. It is defined as the convex hull of the vertices on the chains between v_1, v_2, v_3 together with v_0 and a vertex v_*. The coordinate of v_* is chosen in such a way, that the embedding is submetric. We place v_* on a line given by $x = y = z$. The condition for a submetric embedding is fulfilled if no distance is enlarged. The crucial distances are obtained in the original cube between $p_1, p_{5/4}$ and $p_{3/2}$ and the original corner vertex of the cube p_c. All other distances which occur are symmetric variants of these distances. Therefore we have to choose $v_* = (a, a, a)$ such that the following conditions hold.

$$\begin{array}{rcccl} \|p_1 - p_c\| & = & \sqrt{2}\varepsilon & \geq & \|v_1 - v_c\| \\ \|p_{5/4} - p_c\| & = & \frac{\sqrt{5}}{2}\varepsilon & \geq & \|v_{5/4} - v_c\| \\ \|p_{3/2} - p_c\| & = & \varepsilon & \geq & \|v_{3/2} - v_c\| \end{array}$$

To compute the distances we need the coordinates of $v_1, v_{5/4}$ and $v_{3/2}$ in the specified coordinate system which are

$$\begin{array}{rcl} v_1 & = & (0, \delta, 0), \\ v_{5/4} & = & (\delta\sin(\pi/8), \delta\cos(\pi/8), 0), \\ v_{3/2} & = & \left(\delta/\sqrt{2}, \delta/\sqrt{2}, 0\right). \end{array}$$

We are left with three equation systems which lead to different upper bounds on a. It turns out that the smallest feasible solution for a is obtained by the distance between v_1 and v_*, namely $0.976468\,\varepsilon$. If we set v_* to $(0.9764\,\varepsilon, 0.9764\,\varepsilon, 0.9764\,\varepsilon)$ all distances decrease.

Finally, we have to evaluate the volume of the stars. Each star is divided into tetrahedra. There are two types of tetrahedra, one is given by the convex hull of $v_0, v_1, v_{5/4}, v_*$ and the other by the convex hull of $v_0, v_{5/4}, v_{3/2}, v_*$. Both tetrahedra appear 6 times in every star. That leads to the following expression for the volume of a star;

$$V_{star} = 1.227259706\varepsilon^3.$$

Now we can evaluate the volume of the complete body which is

$$V = V_{corpus} + 12V_{bar} + 8V_{star}.$$

See Figure 4 for the graph of $V(\varepsilon)$ for the feasible values of ε. The volume $V(\varepsilon)$ is maximized at about $\varepsilon_0 = 0.351311$, which induces a volume $V(\varepsilon_0) = 1.2444$. Thus, this improves the bound of Bleecker [2]. The deformed cube for this value of ε is shown in Figure 5. Each star has 3 concave edges depicted as dashed lines. In the next section we refine our construction.

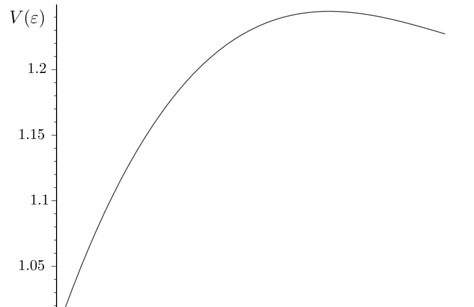

Figure 4: The volume of the deformed cube in terms of ε.

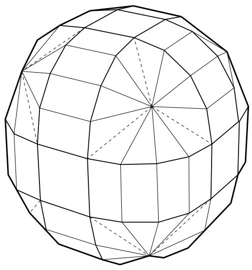

Figure 5: Deformed cube.

3 A Refined Construction

We refine the construction to increase the volume of the cube. A crucial part of the construction was to take two adjacent edges of length ε and turn them into a chain of 4 edges of length $\varepsilon/2$. The deformation puts all vertices of the chain on a quarter circle with radius δ. In the previous section the chain C contained 5 vertices. If we increase the number of vertices on C the deformed cube becomes more "spherish", promising a larger volume. In the limit C is a spherical arc. In the following, we consider this situation.

The value of δ is the radius of a circle with perimeter 8π, therefore

$$\delta = \frac{4\varepsilon}{\pi}.$$

The volume of the corpus is the same as calculated in Section 2. Every bar is a prism with a quarter circle of radius δ as base area. This leads to

$$V_{bar} = \frac{1}{4}(1-2\varepsilon)\pi\delta^2.$$

The stars consist of three equally sized quarter cones. The base area coincides with the base area of the bars. The height of the quarter cones is given by a. The value of a has to be chosen in such a way that the embedding is submetric. We consider the point p_x on the C (See Figure 6). Let its distance from p_1 be x. For the deformed cube we consider the same co-

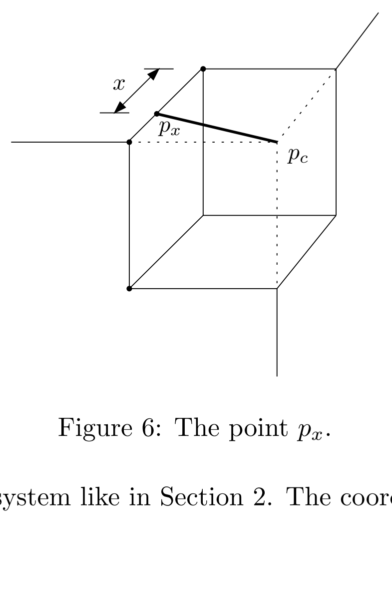

Figure 6: The point p_x.

ordinate system like in Section 2. The coordinates of the point v_x are $(\delta\cos(x/\delta), \delta\sin(x/\delta), 0)$. The distance between p_x and p_c (depicted in Figure 6) equals $\sqrt{(\varepsilon - x)^2 + \varepsilon^2}$. This leads to the following expression for the submetric condition:

$$(\varepsilon - x)^2 + \varepsilon^2 \geq (a - \cos(x/\delta))^2 + (a - \delta\sin(x/\delta))^2 + a^2.$$

The inequality holds with equality if,

$$a(x,\varepsilon) = \frac{1}{3\pi}\Bigg(4\cos\left(\tfrac{x\pi}{4\varepsilon}\right)\varepsilon + 4\sin\left(\tfrac{x\pi}{4\varepsilon}\right)\varepsilon + \sqrt{32\varepsilon^2(\cos\left(\tfrac{x\pi}{4\varepsilon}\right)\sin\left(\tfrac{x\pi}{4\varepsilon}\right) - 1) + 3\pi^2(x^2 + 2\varepsilon^2 - 2x\varepsilon)}\Bigg)$$

The variable a depends on x and ε. We choose x as a multiple of ε. Minimizing the expression over all $x \in [0,1]\varepsilon$ yields a value for a of about $a = 0.9772\ \varepsilon$ which is obtained at about $x = 0.1144\ \varepsilon$. Therefore we can describe the volume of the star by

$$v_{star} = \frac{1}{4}\ 0.9772\ \varepsilon\ \delta^2\pi.$$

Finally we maximize the volume of the whole body (1 corpus, 12 bars, 8 stars) over $\varepsilon \in [0, 0.5]$. It turns out that the maximum is at least 1.2567 which is obtained at about $\varepsilon = 0.37712$.

4 Future Work

Our construction leads to a non-convex body. Due to Alexandrov [1] we know that there exist no convex isometric embedding for a convex polytope with larger volume. It would be interesting to convexify our construction to find a submetric embedding for the cube, which is convex and has largest possible volume. The example given in [6] gives a convex polyhedral construction with a volume only a little large than 1. Related to this question is a conjecture, posed in [6]:

Conjecture 1 (Pak) *Let S_0 be a convex polyhedral surface in $\mathbb{R}^3$ and let S_1 be a convex polyhedral surface submetric to S_0 of greater volume. Then there exists a volume increasing shrinking from S_0 to S_1.*

Since volume-increasing shrinkings exist for polyhedral surfaces in any dimension [6], one can ask for shrinkings for hypercubes. What ratios of the volume can be obtained in dimensions larger than 3? What is the relation between the ratio and the dimension? Notice that if we mimic the construction given in Section 3 in $\mathbb{R}^2$, we end up with a circle, which matches the upper bound.

We have concentrated in our paper on shrinkings of the cube. Our technique is applicable to other surfaces as well. However the computations for other interesting bodies (like platonic solids) still has to be done.

Acknowledgments

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