On the Number of Cycles in Planar Graphs[*]

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Abstract. We investigate the maximum number of simple cycles and
the maximum number of Hamiltonian cycles in a planar graph G with
n vertices. Using the transfer matrix method we construct a family of
graphs which have at least 2.4262^n simple cycles and at least 2.0845^n
Hamilton cycles.

Based on counting arguments for perfect matchings we prove that 2.3404^n
is an upper bound for the number of Hamiltonian cycles. Moreover, we
obtain upper bounds for the number of simple cycles of a given length
with a face coloring technique. Combining both, we show that there is
no planar graph with more than 2.8927^n simple cycles. This reduces the
previous gap between the upper and lower bound for the exponential
growth from 1.03 to 0.46.

1 Introduction

In this paper we consider the following question:

> *How many simple cycles and how many Hamiltonian cycles can there be
> in a planar graph with n vertices?*

Since the determination of the exact numbers seems to be out of reach, our goal
is to learn more about the asymptotic behavior of these numbers. Denoting by
$C_s(G)$ and $C_h(G)$ the numbers of simple cycles and of Hamiltonian cycles in a
graph G we define

$$C_s(n) = \max\left\{C_s(G) \mid G \text{ is a planar graph on } n \text{ vertices}\right\}, \text{ and}$$
$$C_h(n) = \max\left\{C_h(G) \mid G \text{ is a planar graph on } n \text{ vertices}\right\}.$$

It is easy to observe that both $C_s(n)$ and $C_h(n)$ grow exponentially and thus we
are interested in describing this *exponential growth rate* by constants $c, d \in \mathbb{R}$
such that $c^n \leq C_s(n) \leq d^n$, and analogously for $C_h(n)$.

The lower bound $C_s(n) = \Omega(2.259^n)$ was obtained in [1] and is based on counting the number of simple paths connecting two adjacent vertices in a special planar graph on 29 vertices and joining $n/28$ copies of it in a cyclic way. An $O(3.363^n)$ upper bound was proved in the same paper by a probabilistic argument. Here we extend the original problem setting to Hamiltonian cycles.

The problem has gained new attention by some recent results of Sharir and Welzl [2], [3]. They investigate the numbers of several geometric objects on a point set in the plane, among them triangulations and crossing-free spanning cycles. In particular they note that an upper bound on the number of crossing-free spanning cycles can be obtained by combining an upper bound on the number of triangulations with an upper bound on the number of Hamiltonian cycles in planar graphs. Here, we will present the proof to the $\sqrt{6}^n$ upper bound for the number of Hamiltonian cycles, which is quoted in [2] as a personal communication, along with an improvement to $\sqrt[4]{30}^n$.

In [2] Sharir and Welzl prove a bound of $O(86.81^n)$ for the number of crossing-free spanning cycles on n points with an alternative approach. This bound is better than the combined bound and it remains better even if the improved bound for Hamiltonian cycles presented in our paper is used. In fact, the lower bound on the number of Hamiltonian cycles presented in our paper shows that a better combined bound cannot be obtained without improving the bound on the number of triangulations.

The paper is organized as follows: In Section 2 we present new lower bounds for $C_s(n)$ and for $C_h(n)$. We prove $C_s(n) = \Omega(2.4262^n)$ and $C_h(n) = \Omega(2.0845^n)$. Both bounds are based on the so-called transfer matrix method applied on a twisted cylinder.

In Section 3 we prove first the $O(\sqrt[4]{30}^n)$ upper bound on $C_h(n)$. Next we present a new technique for upper bounds on the number of simple cycles with a given length k in planar graphs on n vertices. Combining both we will obtain a new $O(2.8928^n)$ upper bound for $C_s(n)$.

2   Lower bounds

We will present a new lower bound for $C_s(n)$ by counting cycles on the *twisted cylinder*. We use the technique of the *transfer matrix method* (see [4–6]).

The twisted cylinder describes a graph which is parametrized by a *width* w and a *length* l. We will describe the graph by the following construction: Consider an $\lfloor l/w \rfloor \times w$ integer lattice with the upper leftmost point $(0,0)$ and the lower rightmost point (r,w). Furthermore we attach ($l \bmod w$) squares at the right end, starting from the top. As a next step we triangulate each square of the lattice by adding diagonals $((x,y),(x+1,y+1))$ for all appropriate values x and y. Finally we identify all edges $((x,w),(x+1,w))$ with the edges $((x+1,0),(x+2,0))$ for all x smaller than $\lfloor l/w \rfloor$. Observe that this graph is planar since it can be embedded as the graph of a 3-polytope. Figure 1 shows a twisted cylinder of length 41 and width 5, and Figure 2 shows a planar embedding of a twisted cylinder of length 12 and width 6. To count the cycles, we construct the cylinder
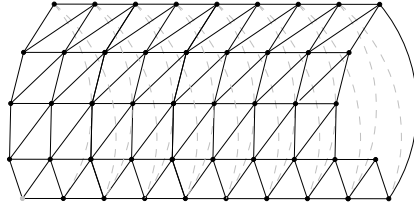
Fig. 1. The graph of a twisted cylinder.

by increasing its length consecutively. We name the cylinder of width w after k rounds Z_k^w and call the last inserted $w + 1$ points its *border*. During the construction we have to deal with unfinished cycles. These cycles are represented as non-intersecting paths which start and end at the border of Z_k^w. To complete a cycle we need the information which of the points at the border belong to the same path. We will store this information in a string of length $w + 1$ which we call the *signature*. The last point introduced corresponds to the first character of the signature, its predecessor to the second character and so on. Every path has a start and an end point on the border. The point that was introduced later is considered as the start point, the other as the end point of a path. We associate a start point at the border with an A. The position of the end point of a path will be marked in the signature as B. Interior points of paths at the border are denoted by X. A point at the border that is not used from any path will be represented as O in the signature. Thus we get as signature a string from $\{A, B, X, O\}^{w+1}$. Figure 2 shows an example which has the signature $AXOAOBB$. Notice that
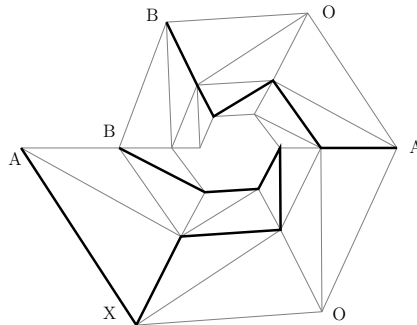

Fig. 2. Partially constructed cycle with signature $AXOAOBB$.

the signatures can be represented as 2-Motzkin paths [7, Exercise 6.38.], which are one of the numerous incarnations of Catalan structures.

We come back to the counting procedure. During the construction we trace the number of completed and uncompleted cycles. We count the different ways of generating an uncompleted cycle by a variable indexed by its signature. The completed cycles are stored in a distinct variable. All variables are stored in a vector which we call the *state vector S_k*.

Going from cylinder Z_k^w to Z_{k+1}^w will change the state vector. We introduce three new edges and therefore have at most 7 ways to continue uncompleted cycles (choosing all 3 edges will not give a valid successor state). Not all of these choices will produce a valid signature for the successor state. For example the signature $AXOAOBB$ (Figure 2) has four successors, depicted in Figure 3.
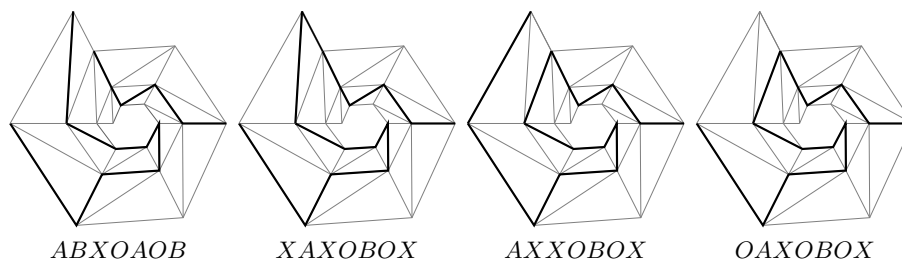

$ABXOAOB$ $XAXOBOX$ $AXXOBOX$ $OAXOBOX$

Fig. 3. The successor states from Figure 2.

It is not hard to see that every entry of the "new" state vector S_{k+1} is a non-negative linear combination of the entries of the "old" state vector S_k. Thus $S_{k+1} = T \cdot S_k$, where T is a square matrix with non-negative entries, which is called the *transfer matrix*. By construction this matrix T does not depend on k, and thus for every $k \geq 0$ we have $S_k = T^k \cdot S_0$. The entry in S_k for the completed cycles in Z_k^w is then a lower bound for $C_s(k + w)$.

This entry can be represented as $e^t \cdot S_k$ for the appropriate unit basis vector e. Thus we are interested in the asymptotic behavior of a sequence with elements $p_k = a^t \cdot T^k \cdot b$, where a and b are vectors and T is a square matrix. By considering the Jordan canonical form $T = X^{-1} \cdot J \cdot X$ it is easy to see that $p_k = O(q(k)c^k)$, where c is the largest absolute value of any eigenvalue of T and q is a polynomial of degree smaller than the multiplicity of any eigenvalue of maximal absolute value.

If in our case we remove all unreachable configurations from consideration then the resulting transfer matrix T will be primitive in the sense that for some $\ell > 0$ all entries of T^ℓ are strictly positive. In this case the Perron-Frobenius Theorem [8] guarantees that the eigenvalue of largest absolute value is real and unique. Thus $S_k = O(c^k)$, where c is the largest real eigenvalue of the transfer matrix T.

The generation of T and the computation of its eigenvalues was done with the help of computer programs. We omit the details of the computation of T. The

correctness of the calculations was checked by two independent implementations. We observed that larger widths will result in larger growth. The largest width our implementations could handle is 13. The largest absolute eigenvalue for T was computed as 2.4262. The computation was done on a AMD Athlon 64 with 1.8 GHz and 1 GB of RAM. It required 5 days of CPU-time and 550 MB of memory. The implementation can be found under [9]. The results of the computation are listed in Table 1.

Theorem 1. *The maximal number of simple cycles in a planar graph G with n vertices is bounded from below by $\Omega(2.4262^n)$.*

We can also construct a lower bound for Hamiltonian cycles with the method from above. To this end we restrict the state transitions in such a way that if a vertex vanishes from the border, it is guaranteed to be on some path. We forbid all sequences which contain a O as character and calculate the modified transfer matrix.

The largest eigenvalue for the modified transfer matrix is 2.0845. It is obtained for a twisted cylinder of width 13. See Table 1 for the results of the computation.

w	λ_T H. cyc.	λ_T simple cyc.	w	λ_T H. cyc.	λ_T simple cyc.
2	1.8124	1.9659	8	2.0688	2.4078
3	1.9557	2.2567	9	2.0740	2.4139
4	2.0022	2.3326	10	2.0777	2.4183
5	2.0335	2.3654	11	2.0805	2.4217
6	2.0507	2.3858	12	2.0827	2.4242
7	2.0614	2.3991	13	2.0845	2.4262

Table 1. Eigenvalues λ_T of the transfer matrix T, generated for Hamiltonian cycles (H. cyc.) and simple cycles (simple cyc.) depending on the width of the twisted cylinder.

Theorem 2. *The maximal number of Hamiltonian cycles in a planar graph G with n vertices is bounded from below by $\Omega(2.0845^n)$.*

3 Upper bounds

3.1 Hamiltonian Cycles

In this section G denotes a planar graph with n vertices, e edges, and f faces. Since additional edges cannot decrease the number of cycles, we focus on triangulated planar graphs. In this case we have $3f = 2e$, which leads to $e = 3n - 6$ and $f = 2n - 4$.

Let us assume first that n is even and let $M(G)$ denote the number of perfect matchings in G. By a theorem of Kasteleyn, c.f. [10], there is an orientation of

the edges of G such that the corresponding skew symmetric adjacency matrix A characterizes $M(G)$ in the following way:

$$(M(G))^2 = |\det(A)|.$$

Note that all but $6n - 12$ entries of A are zero and the nonzero entries are 1 or -1. In this situation we can apply the Hadamard bound for determinants and we obtain $|\det(A)| \leq \sqrt{6}^n$.

In this way we obtain an $\sqrt[4]{6}^n$ upper bound on the number of perfect matchings in G, which improves the $O(\sqrt{3}^n)$ bound from [11]. Moreover, our bound can be improved for graphs with few edges.

Theorem 3. *The number of perfect matchings in a planar graph G with n vertices is bounded from above by $\sqrt[4]{6}^n$.*
The number of perfect matchings in a planar graph G with n vertices and at most kn edges is bounded from above by $\sqrt[4]{2k}^n$.

Our first bound on the number of Hamiltonian cycles follows from Theorem 3 by an easy observation.

Theorem 4. $C_h(n) = O\left(\sqrt[4]{30}^n\right) = O\left(2.3404^n\right).$

Proof. Any Hamiltonian cycle in a graph G with an even number of vertices splits into two perfect matchings, which implies $C_h(G) \leq (M(G))^2 \leq \sqrt{6}^n$. The following modification of the arguments above results in a slight improvement of that bound:

Splitting a Hamiltonian cycle into two perfect matchings, we fix the matching with the lexicographically smallest edge as the first matching m_1 and the other one as the second matching m_2. It follows that if m_1 is fixed, m_2 is a matching in a graph with $2.5\, n - 6$ edges. Repeating the above observations for both matchings, we get

$$M_1(G) \leq \sqrt[4]{6}^n, \qquad M_2(G) \leq \sqrt[4]{5}^n \quad \text{and together} \quad C_h(G) \leq \sqrt[4]{30}^n.$$

Finally we study the case that n is odd. We choose in G a vertex v of degree at most 5, and for each e incident to v we consider the Graph G_e obtained from G by contracting e. Any Hamiltonian cycle in G contains two edges e and e' incident to v and hence induces a Hamiltonian cycle in G_e and $G_{e'}$. On the other hand, any Hamiltonian cycle in some G_e may be extended in up to two ways to a Hamiltonian cycle in G. Thus we obtain an upper bound on $C_h(G)$ by adding the number of Hamiltonian cycles in the at most five planar graphs G_e, leading to a bound of $C_h(G) \leq 5\sqrt[4]{30}^{n-1}$. $\qquad\square$

3.2 Simple Cycles

We start with a new upper bound for the number of cycles in planar graphs and successively improve the bound.

Instead of counting cycles we count paths on G, which can be completed to a simple cycle. We call these paths *cycle-paths*. Their number is an upper bound

for the number of cycles. The number of cycle-paths is maximized when G is triangulated. Therefore we assume that G is triangulated.

There exist n paths of length 0. The number of all cycle-paths in G of nonzero length is at most the number of edges e times the maximum number of cycle-paths starting from an arbitrary edge. Thus the exponential growth of the number of cycle-paths is determined by the number of cycle-paths starting from an edge.

Lemma 1. *The maximum number of cycle-paths on G starting from an edge is bounded by $O(n) \cdot 3^n$.*

Proof. We give the starting edge an orientation. We consider only paths in the direction induced by this orientation. The total number of cycle-paths starting from this edge is at most twice the number of cycle-paths with the chosen orientation.

We associate cycle-paths with the nodes of a tree. The root of the tree contains the path of length one corresponding to the starting edge. The children of a tree node contain paths starting with the path stored in the predecessor plus an additional edge. Every cycle-path is only stored in one tree node.

Every cycle-path in G corresponds to a partial red-blue coloring of the faces of G. The coloring is defined as follows: The faces right of the oriented path will be colored blue the faces left of the oriented path red (see Figure 4). We color all faces incident to an inner vertex or the starting edge of the path. The coloring is consistent, because we consider only paths which can be extended to cycles. Therefore the colors correspond to a part of the interior or exterior region induced by the cycle.
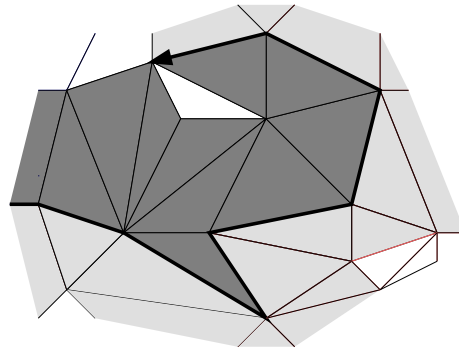

Fig. 4. Example of an induced red-blue coloring by a path (light gray corresponds to red, dark gray to blue).

We construct the tree top down. When we enter a new tree node, the color of at least two faces incident to the last vertex v_i of the path is given. It might be

that other faces incident to v_i have been colored before. In that case we color the faces incident to v_i which lie in between two red faces red. The faces which are located in between two blue faces will be colored blue. Observe that at most one non-colored connected region incident to v_i remains. Otherwise it is not possible to extend the path to a cycle and therefore the path stored in this tree node is not a cycle-path. Figure 5 illustrates this procedure. Let k_v be the number of
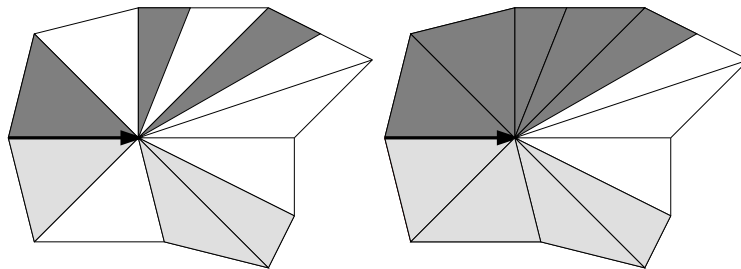

Fig. 5. Completing the red-blue coloring when entering a new vertex (light gray corresponds to red, dark gray to blue).

faces of the non-colored region incident to v. We have $k_v + 1$ different ways to continue the path and therefore $k_v + 1$ children of its tree node. No matter which child we choose, we will color all faces incident to v.

It remains to analyze the number of nodes in the tree. A bound on the number of nodes can be expressed by the following recurrence:

$$P(n, f) \leq (k_v + 1)P(n - 1, f - k_v) + 1.$$

Because we want to maximize the number of nodes in the tree, we can assume that the k_vs for all v within a level l of the tree are equal. This holds for the root and by an inductive argument for the whole tree. Let κ_l denote the number k_v for the vertices v on level l.

$P := P(n - 2, 2n + 2)$ will give us the number of nodes in the tree. We know that $P(0, \cdot) = P(\cdot, 0) = 1$. All κ_ls have to be non-negative numbers. The κ_l along a path of length L have to fulfill the condition $\sum_{l \leq L} \kappa_l \leq 2n + 2$. A path is of length at most $n - 2$, and therefore we can bound P by

$$1 + \sum_{L=1}^{n-2} \prod_{l \leq L} (\kappa_l + 1). \tag{1}$$

We are interested in a set κ_l which maximizes (1). Due to the convexity of (1) the maximum will be obtained when all κ_l are equal. Thus (1) is bounded by $1 + \sum_{i \leq n} (\frac{2n+2}{n-2} + 1)^i$. Therefore the exponential growth of the maximum number of cycle-paths is $O(n) \cdot 3^n$. $\qquad \square$

This already yields an improvement of the best known upper bound for cycles in planar graphs.

Observation 1. *The number of simple cycles on a planar graph with n vertices is bounded from above by $O(n) \cdot 3^n$.*

We improve the obtained upper bound further. For this we go back to the proof of Lemma 1. Instead of considering cycle-paths of length n, we focus on shorter cycle-paths of length αn, where $\alpha \in [0, 1]$.

Lemma 2. *Let $C_s^\alpha(n)$ be the number of simple cycles of length αn in a planar graph with n vertices and f faces. Then we have*

$$C_s^\alpha(n) \leq \left(\frac{f}{\alpha n} + 1 \right)^{\alpha n}. \tag{2}$$

Proof. We reconsider the argumentation which led to Observation 1 and notice that $P(k, f)$ will be maximized by equally distributed values of κ_l. Therefore we set $\kappa_l = f/(\alpha n)$, which proves the Lemma. $\qquad \square$

As final step we combine the result from Lemma 2 with the results of Section 3.1.

Theorem 5. *The number of simple cycles in a planar graph G with n vertices is bounded from above by $O(2.89278^n)$.*

Proof. An upper bound ν^n for the number of Hamiltonian cycles will always imply an upper bound for $C_s(G)$ since every simple cycle is an Hamiltonian cycle on a subgraph of G. This leads to

$$C_s(n) \leq \sum_{t \leq n} \binom{n}{t} \nu^t = (1 + \nu)^n.$$

Plugging in our bound of $\sqrt[4]{30}$ for ν yields $C_s(n) \leq 3.3404^n$, which is larger than 3^n. Responsible for this are cycles with small length. When choosing a small subset of vertices, it is unlikely that they are connected in G. Therefore the Hamiltonian cycles counted for this subset will not correspond to cycles in G. Thus we overestimate the number of small cycles.

We modify the upper bound induced by the Hamiltonian cycles such that they can express $C_s^\alpha(n)$. Every αn-cycle is a Hamiltonian cycle on a subgraph of size αn. Thus

$$C_s^\alpha(n) \leq \binom{n}{\alpha n} \nu^{\alpha n} \leq 5 \binom{n}{\alpha n} (\sqrt[4]{30})^{\alpha n}$$

Since $\sum_i \binom{n}{i} \alpha^i (1-\alpha)^{n-i} = 1$, for $0 \leq \alpha \leq 1$ every summand of this sum is at most 1. Considering the summand for $i = \alpha n$ yields $\binom{n}{\alpha n} \leq (1/(\alpha^\alpha (1-\alpha)^{1-\alpha}))^n$ and therefore

$$C_s^\alpha(n) = O\left(\left(\frac{\sqrt[4]{30}^\alpha}{\alpha^\alpha (1 - \alpha)^{(1-\alpha)}} \right)^n \right). \tag{3}$$

So far we know two upper bounds for $C_s^\alpha(n)$. The two bounds are shown in Figure 6. The graph of (3) is represented as dashed gray curve, whereas the graph of (2) is depicted solid black. Clearly the maximum of the lower envelope of the two functions induces an upper bound for the exponential growth of cycles in G.

One can observe that the two functions intersect in the interval $[0, 1]$ in only one point, which is approximately $\tilde{\alpha} = 0.91925$. The maximal exponential growth is realized for this α. Evaluating $C_s^{\tilde{\alpha}}(n)$ yields a bound of 2.89278^n on the number of cycles. $\qquad\square$

At its core the bound of $\left(\frac{f}{\alpha n} + 1\right)^{\alpha n}$ in Lemma 2 comes from *consuming f* faces in αn steps. A similar bound can be obtained by consuming edges instead. In this case we do not need a coloring scheme. In each step we get as many ways to continue the cycle-path as the number of edges consumed in the step. This yields a bound of $\left(\frac{e}{\alpha n}\right)^{\alpha n}$. For $e = 3n - 6$, $f = 2n - 4$, and $\alpha < 1$ the bound obtained by considering faces is stronger.

For this counting argument the graph does not need to be planar. With $\alpha = 1$ it yields a bound of $\left(\frac{e}{n}\right)^n$ on the number of cycles in the graph. This bound has been independently observed by Sharir and Welzl [2].
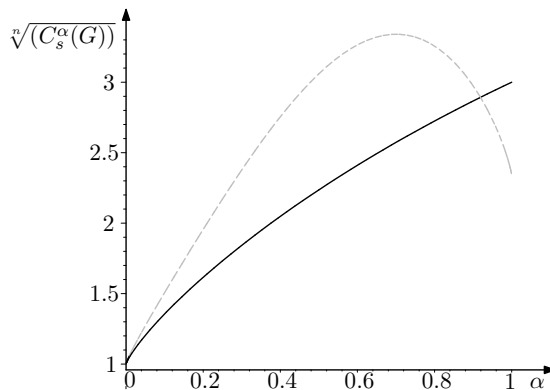

Fig. 6. Plot of the two bounds for $C_s^\alpha(G)$.

4   Discussion

We improved the lower and upper bounds for the number of simple cycles in planar graphs. This reduces the gap between the upper and lower bound for the exponential growth from 1.03 to 0.46. We believe that the truth is closer to the lower bound. This is indicated by the technique sketched in the following which might further improve the upper bound.

For Observation 1 the worst case scenario is the situation where there are three possible ways to continue the cycle-path. However it is clear that this situation will not constantly occur during the construction of the cycle-paths. To use this fact we compute recurrences for the number of cycle-paths by simultaneously analyzing two or more consecutive levels of the tree which stores the cycle-paths. A careful analysis reveals other effects in this setting which also reduce the number of cycle-paths. In particular, vertices and faces will be surrounded and absorbed by the colored regions. The main part of the analysis is an intricate case distinction for which we have not checked all cases yet.

Furthermore we used the transfer matrix approach on the twisted cylinder to obtain lower bounds for other structures (for instance perfect matchings) on planar graphs. Moreover we adapted the counting procedure for sub-classes of planar graphs (for instance grid graphs). The results of these computations have not yet been double-checked and we therefore do not include them in this extended abstract.

Acknowledgments

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