# Inflating the Cube by Shrinking

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## ABSTRACT

We present a continuous submetric deformation of the surface of the cube which increases the enclosed volume by about 25.67%.

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**General Terms:** Theory
**Keywords:** Submetric embeddings, surface deformation

## 1. INTRODUCTION

Every polyhedral surface in $\mathbb{R}^3$ has a volume-increasing isometric deformation [7]. However little is known about isometric embeddings maximizing the volume. For the polyhedra typically studied, i.e., platonic solids and doubly covered two-dimensional shapes, there is a large gap between the lower and upper bound for the maximum enclosed volume. We address the problem of how large the volume of a body with a surface isometric to that of a cube can be.

An embedding $h : S \to \mathbb{R}^3$ of a surface $S$ in $\mathbb{R}^3$ is *isometric* (resp. *submetric*) if the length of any rectifiable curve in $S$ is constant (resp. non-increasing) under $h$. An isometric (resp. submetric) deformation is a continuous map $H : S \times [0, \varepsilon_0] \to \mathbb{R}^3$ such that $h_\varepsilon(\cdot) := H(\cdot, \varepsilon)$ is an isometric (resp. submetric) embedding for all $0 \leq \varepsilon \leq \varepsilon_0$. Instead of isometric embeddings, submetric embeddings can be used since every submetric embedding of a polyhedral surface has an isometric embedding arbitrary close to it [4, 7].

The idea of considering volume-increasing isometric embeddings is due to Bleecker [2]. He proves that a volume-increasing continuous isometric deformation exists for every simplicial convex surface in $\mathbb{R}^3$. By Alexandrov's uniqueness theorem [1] a body resulting from such a deformation must be non-convex. By Bellows conjecture (proved in [5]) the deformation does not preserve the faces of the polytope.

Recently, Pak [8] gave an easy construction for increasing the volume of the unit cube to about 1.1812 based on the work of Milka [6]. A more involved construction of Bleecker [2] yields a volume of about 1.2187. A simple upper bound of about 1.3820 can be obtained by the volume of the sphere which has the same surface area as the cube.

In this paper, we present a *shrinking*, i.e., a continuous, submetric deformation of the unit cube for which the resulting volume is at least 1.2567. We first present a simple construction which we then refine. This also improves the lower bound on the volume of a surface isometric to that of a unit cube. Detailed calculations for the constructions in this paper can be found in [3]. The idea of looking at shrinkings in order to get isometric embeddings is due to Pak [7].

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## 2. FIRST CONSTRUCTION

We present volume-increasing submetric embeddings of the cube. The embeddings are parametrized by $\varepsilon \in [0, 0.5]$. Increasing $\varepsilon$ from 0 yields a continuous deformation. We first present a construction with volume about 1.2444, which we improve in the next section.

Our approach is a refinement of Igor Pak's work [7, 8]. The original cube is given as the convex hull of the set $\{0, 1\}^3$. We denote vertices on the surface of the cube by $p_i$. The same vertex in the deformed cube is denoted as $v_i$.

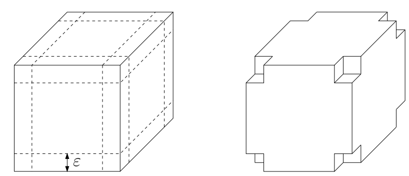


**Figure 1: Cutting off $\varepsilon$ cubes.**

First we cut off $\varepsilon$-cubes at every corner of the cube (see Figure 1). Next we deform the remaining part of the cube. We place one vertex in the middle of every $\varepsilon$ segment (see Figure 2.a). The segments between $p_1, p_{5/4}, p_{3/2}, p_{7/4}, p_2$ have the length $\varepsilon/2$. Let the framework induced by this chain be $C$. We move the vertices of $C$ such that $v_1, \ldots, v_2$ lie on a quarter-circle (see Figure 2.b). We apply the deformation for all corresponding pairs of $\varepsilon$ segments. This leads to a body that we divide into a *corpus* and 12 *bars*. The 8

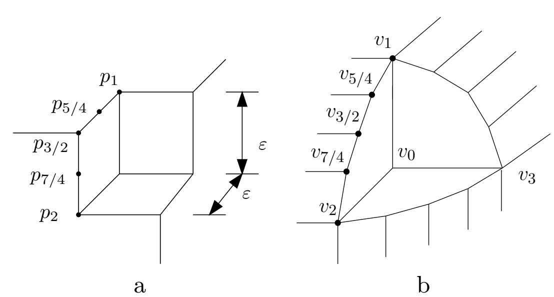


**Figure 2: Bending the chains induced by a pair of $\varepsilon$ segments.**

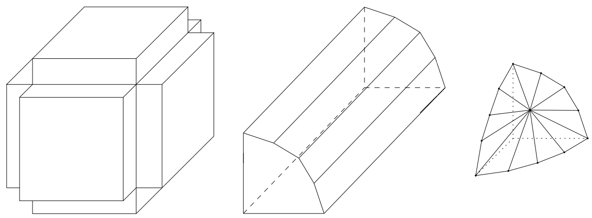

**Figure 3: Corpus and bar and star.**

cut off parts are called *stars*. Figure 3 shows the deformed parts. After the deformation a bar is a prism with a 6-gon as base area. The 6-gon is inscribed in quarter circle. Its shorter edges have the length $\varepsilon/2$. The radius of the quarter circle is denoted by $\delta$. The volume of one bar is given by

$$V_{bar} = 2(1-2\varepsilon)\delta^2 \sin(\pi/8).$$

The corpus is the remaining part after cutting off the bars. Its volume is given by

$$V_{corpus} = (1-2\varepsilon)^3 \; + \; 6\delta(1-2\varepsilon)^2.$$

It remains to place the deformed stars at the corners of the body. We have to deform the $\varepsilon$-cubes, such that they fit into the open 12-gons formed by three chains $C$ of the body. Consider the open part depicted in Figure 2.b. The deformed star is the convex hull of the vertices on the chains between $v_1, v_2, v_3$ together with $v_0$ and a vertex $v_*$. The coordinate of $v_*$ is chosen in such a way, that the embedding is submetric. We place $v_*$ on a line given by $x = y = z$. This yields coordinates $v_* = (0.9764\,\varepsilon, 0.9764\,\varepsilon, 0.9764\,\varepsilon)$.

Finally, we have to evaluate the volume of the stars. Each star is divided into tetrahedra. There are two types of tetrahedra, one is given by the convex hull of $v_0, v_1, v_{5/4}, v_*$ and the other by the convex hull of $v_0, v_{5/4}, v_{3/2}, v_*$. Both tetrahedra appear 6 times in every star. That leads to a volume of a star of

$$V_{star} = 1.227259706\varepsilon^3.$$

Now we can evaluate the volume of the complete body which is

$$V = V_{corpus} + 12V_{bar} + 8V_{star}.$$

See Figure 4 for the graph of $V(\varepsilon)$ for feasible values of $\varepsilon$.

The volume $V(\varepsilon)$ is maximized at about $\varepsilon_0 = 0.351311$, which yields a volume $V(\varepsilon_0) = 1.2444$. This improves the bound of Bleecker [2]. The deformed cube for this value of $\varepsilon$ is shown in Figure 5. Each star has 3 concave edges depicted as dashed lines.

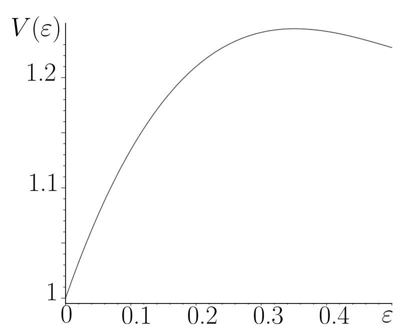


**Figure 4: The volume of the deformed cube.**

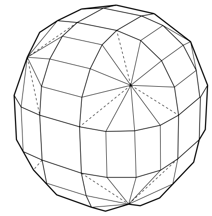

**Figure 5: Deformed cube.**

# 3. A REFINED CONSTRUCTION

We refine the construction to increase the volume of the cube. A crucial part of the construction was to take two adjacent edges of length $\varepsilon$ and turn them into a chain of 4 edges of length $\varepsilon/2$. The deformation puts all vertices of the chain on a quarter circle with radius $\delta$. In the previous section the chain $C$ contained 5 vertices. If we increase the number of vertices on $C$ the deformed cube becomes more "spherish", promising a larger volume. In the limit $C$ is a spherical arc. In the following, we consider this situation.

The value of $\delta$ is the radius of a circle with perimeter $8\pi$, thus $\delta = 4\varepsilon\pi$. The volume of the corpus is the same as calculated in Section 2. Every bar is a prism with a quarter circle of radius $\delta$ as base area. This leads to

$$V_{bar} = (1-2\varepsilon)\pi\delta^2/4.$$

The stars consist of three equally sized quarter cones. The base area coincides with the base area of the bars. The height of the quarter cones is given by $a$. The value of $a$ has to be chosen in such a way that the embedding is submetric.

As feasible height we get $a = 0.9772\,\varepsilon$. We get

$$V_{star} = 0.9772\;\varepsilon\;\delta^2\pi/4.$$

Maximizing the volume of the whole body (1 corpus, 12 bars, 8 stars) over $\varepsilon \in [0, 0.5]$ leads to a volume of 1.2567 obtained at about $\varepsilon = 0.37712$.

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