Knightian Robustness of Single-Parameter Domains

Alessandro Chiesa Silvio Micali Zeyuan Allen Zhu

MIT

March 25, 2014

Abstract

We consider players that have very limited knowledge about their own valuations. Specifically, the only information that a *Knightian player* i has about the profile of true valuations, θ^*, consists of a set of distributions, from one of which θ_i^* has been drawn.

We prove a "robustness" theorem for Knightian players in single-parameter domains: every mechanism that is weakly dominant-strategy truthful for classical players continues to be well-behaved for Knightian players that choose undominated strategies.

1 Introduction

In [CMZ14] we motivate the problem of mechanism design for Knightian players, and prove that (1) dominant-strategy mechanisms for single-good and multi-unit auctions cannot provide good social-welfare efficiency, but (2) the second-price and Vickrey mechanisms deliver good social-welfare performance, for these two settings, in undominated strategies.

In this report, we prove a "robustness" theorem for single-parameter domains. Namely, consider a mechanism M for a single-parameter domain and suppose that M, when players have perfect information about their own valuations, is weakly dominant-strategy truthful. Now consider the same mechanism M, but with Knightian players that, not having any dominant strategy to play, choose to play undominated strategies. We prove that the set of undominated strategies is well-behaved, in the sense that these strategies do not deviate from the players' approximate information about his own valuation.

2 Model

In a classical single-parameter domain, there is a set $\mathcal{A}$, the set of all possible allocations; for each player i there exists a publicly known subset $\mathcal{S}_i \subseteq \mathcal{A}$; and the set of possible valuations for player i, Θ_i, consists of all functions mapping $\mathcal{A}$ to the reals, subject to the following constraints: for each $\theta_i \in \Theta_i$,

(1) $\theta_i(x) = 0 \quad \forall x \notin \mathcal{S}_i$ and

(2) $\theta_i(x) = \theta_i(y) \quad \forall x, y \in \mathcal{S}_i$.

We denote the true valuation of player i by θ_i^*.

(The term "single-parameter" derives from the fact that each $\theta_i \in \Theta_i$ coincides with a single number: i's value for, say, the lexicographically first element of $\mathcal{S}_i$. The term "classical" emphasizes that each player knows exactly his own true valuation.)

The set of possible outcomes is $\Omega \stackrel{\text{def}}{=} \mathcal{A} \times \mathbb{R}^n_{\geq 0}$. If $(A, P) \in \Omega$, we refer P_i as the price charged to player i. We assume quasi-linear utilities. That is, the utility function U_i of a player i maps a valuation θ_i and an outcome $\omega = (A, P)$ to $U_i(\theta_i, \omega) \stackrel{\text{def}}{=} \theta_i(A) - P_i$.

If ω is a distribution over outcomes, we also denote by $U_i(\theta_i, \omega)$ the expected utility of player i.

Single-parameter domains are general enough to include several settings of interest: in particular, provision of a public good[1] [Cla71], bilateral trades [MS83], and buying a path in a network [NR01].

2.1 Knightian Valuation Uncertainty

In our model, a player i's sole information about θ^* consists of $\mathcal{K}_i$, a set of distributions over Θ_i, from one of which θ_i^* has been drawn. (The true valuations are uncorrelated.) That is, $\mathcal{K}_i$ is i's sole (and private) information about his own true valuation θ_i^*. Furthermore, for every opponent j, i has no information (or beliefs) about θ_j^* or $\mathcal{K}_j$.

Given that all he cares about is his expected (quasi-linear) utility, a player i may 'collapse' each distribution $D_i \in \mathcal{K}_i$ to its expectation $\mathbb{E}_{\theta_i \sim D_i}[\theta_i]$.[2] Therefore, for single-parameter domains, a *mathematically equivalent* formulation of the Knightian valuation model is the following:

Definition 2.1 (Knightian valuation model)**.** *For each player i, i's sole information about θ^* is a set K_i, the* candidate (valuation) set of i*, such that $\theta_i^* \in K_i \subset \Theta_i$.*

We refer to an element of K_i as a candidate valuation*.*

In Knightian valuation model, a mechanism's performance will of course depend on the inaccuracy of the players' candidate sets, which we measure as follows.

Definition 2.2. *Let $K_i^\perp \stackrel{\text{def}}{=} \inf K_i$ and $K_i^\top \stackrel{\text{def}}{=} \sup K_i$.*

The candidate set K_i of a player i is (at most) δ-approximate *if $K_i^\top - K_i^\perp \leq \delta$.*

A single-parameter domain is (at most) δ-approximate *if each K_i is δ-approximate.*

[1] Indeed, in the provision of a public good, $\mathcal{A}$ has just two elements, a (i.e., the good is provided), which different players may value differently, and b (i.e., the good is not provided), which all players value 0.

[2] Whatever the auction mechanism used, this equivalence holds for any auction where each Θ_i is a *convex* set. In particular, this includes unrestricted combinatorial auctions of m distinct goods.

2.2 Social Welfare, Mechanisms, and Knightian Dominance

Social welfare. The social welfare of an allocation $A \in \mathcal{A}$, $\mathrm{SW}(A)$, is defined to be $\sum_i \theta_i^*(A)$; and the maximum social welfare, MSW, is defined to be $\max_{A\in\mathcal{A}} \mathrm{SW}(A)$. (That is, SW and MSW continue to be defined relative to the players' true valuations θ_i^*, whether or not the players know them exactly.)

More generally, the social welfare of an allocation A relative to a valuation profile θ, $\mathrm{SW}(\theta, A)$, is $\sum_i \theta_i(A)$; and the maximum social welfare relative to θ, $\mathrm{MSW}(\theta)$, is $\max_{A\in\mathcal{A}} \mathrm{SW}(\theta, A)$. Thus, $\mathrm{SW}(A) = \mathrm{SW}(\theta^*, A)$ and $\mathrm{MSW} = \mathrm{MSW}(\theta^*)$.

General mechanisms and strategies. A mechanism M specifies, for each player i, a set S_i. We interchangeably refer to each member of S_i as a pure *strategy/action/report* of i, and similarly, a member of $\Delta(S_i)$ a mixed strategy/action/report of i.

After each player i, simultaneously with his opponents, reports a strategy s_i in S_i, M maps the reported strategy profile s to an outcome $M(s) \in \Omega$.

If M is probabilistic, then $M(s) \in \Delta(\Omega)$. Thus, as per our notation, $U_i(\theta_i, M(s)) \stackrel{\text{def}}{=} \mathbb{E}_{\omega\sim M(s)}[U_i(\theta_i, \omega)]$ for each player i.

Note that $S_i = \Theta_i$ for the direct mechanisms in the classical setting, but may be arbitrary in general.

Knightian undominated strategies. Given a mechanism M, a pure strategy s_i of a player i with a candidate set K_i is *(weakly) undominated*, in symbols $s_i \in \mathsf{UD}_i(K_i)$, if i does not have another (possibly mixed) strategy σ_i such that

(1) $\forall \theta_i \in K_i\ \forall s_{-i} \in S_{-i} \quad \mathbb{E}U_i\big(\theta_i, M(\sigma_i, s_{-i})\big) \geq U_i\big(\theta_i, M(s_i, s_{-i})\big)$, and

(2) $\exists \theta_i \in K_i\ \exists s_{-i} \in S_{-i} \quad \mathbb{E}U_i\big(\theta_i, M(\sigma_i, s_{-i})\big) > U_i\big(\theta_i, M(s_i, s_{-i})\big)$.

If K is a product or a profile of candidate sets, that is, if $K = (K_1, \ldots, K_n)$ or $K = K_1 \times \cdots \times K_n$, then $\mathsf{UD}(K) \stackrel{\text{def}}{=} \mathsf{UD}_1(K_1) \times \cdots \times \mathsf{UD}_n(K_n)$.

Note that the above notion of an undominated strategy is a natural extension of its classical counterpart, but other extensions are possible.

Weakly dominant-strategy truthfulness in classical settings. Finally, let us recall what it means for a mechanism M to be weakly dominant-strategy truthful (weakly DST) when every player i knows θ_i^* exactly. Namely, for each player i:

(0) $S_i = \Theta_i$

(1) $\forall v_i \in \Theta_i \; \forall v_i' \in \Theta_i \; \forall v_{-i} \in \Theta_{-i} \quad U_i\big(v_i, M(v_i, v_{-i})\big) \geq U_i\big(v_i, M(v_i', v_{-i})\big)$

(2) $\forall v_i \in \Theta_i \; \forall v_i' \in \Theta_i \setminus \{v_i\} \; \exists v_{-i} \in \Theta_{-i} \quad U_i\big(v_i, M(v_i, v_{-i})\big) > U_i\big(v_i, M(v_i', v_{-i})\big)$.

(For comparison, the notion of a DST mechanism omits the last condition above.)

3 Result

We prove the Knightian robustness of many mechanisms at once as follows.

Theorem 1. *Let M be a weakly dominant-strategy truthful mechanism for classical single-parameter domains. Then, in this domain with Knightian valuation uncertainty, for every player i, $\mathsf{UD}(K_i) \subseteq \big[K_i^{\perp}, K_i^{\top}\big]$.*

Discussion. The above theorem implies that the behavior of (weakly dominant-strategy truthful) mechanisms in a δ-approximate single-parameter domains gracefully degrades with δ. In particular, it implies that, when applied to the provision of a public good in the presence of n Knightian players, the VCG mechanism guarantees, in undominated strategies, a social welfare $\geq \text{MSW} - 2n\delta$. As another example, when applied to buying paths in a network, the VCG mechanism guarantees a social welfare $\geq$ MSW $- 2m\delta$, where m is the number of edges in the network. Finally, we note that the proof of Theorem 1 easily extends to imply an analogous result for the VCG mechanism for *single-minded combinatorial auctions*, which are not quite single-parameter domains.[3]

More generally, Theorem 1 implies that, for all weakly dominant-strategy mechanisms M (which include those of [Cla71, MS83, NR01])

'the outcome $M(v)$ is sufficiently good

whenever $\max_i |v_i - \theta_i^*|$ *is sufficiently small for all i and $\theta_i^* \in K_i$'.*

[3]In such an auction, there are m distinct goods, and each player i values, positively and for the same amount θ_i^*, only the supersets of a given subset $\mathcal{S}_i$ of the goods. This auction is not single-parameter because $\mathcal{S}_i$ is *private*, that is, known solely to i. Accordingly, i's true valuation can be fully described only by the number θ_i^* *and* the subset $\mathcal{S}_i$. The VCG mechanism for single-minded auctions ensures, in undominated strategies, a social welfare that is at least MSW $- 2\min\{n, m\}\delta$.

Proof. The theorem is obvious when $K_i = \{\theta_i^*\}$ is a singleton: since reporting the truth is a weakly dominant strategy, it dominates all other strategies so that $\mathsf{UD}(K_i) = \{\theta_i^*\}$ must also be a singleton. For the rest of the proof we assume that K_i has at least two distinct valuations.

We begin by recalling the following fact about dominant-strategy truthful mechanisms in single-parameter domains where each player perfectly knows his own true valuation [AT01]:

> Let M be a mechanism for a single-parameter domain, and let $f_i(v) \in [0,1]$ be the probability that the allocation chosen by M, under strategy profile v, is in player i's set $\mathcal{S}_i$. Then, M is dominant-strategy truthful if and only if (a) f is monotonically non-decreasing, i.e., $f_i(v_i, v_{-i}) \leq f_i(v_i', v_{-i})$ whenever $v_i \leq v_i'$, and (b) player i's expected price on input v, denoted by $p_i(v)$, equals to $v_i \cdot f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(z, v_{-i})\, dz$.

Having recalled the above fact, we now prove that, for any Knightian player i with candidate set $K_i = [K_i^\perp, K_i^\top]$,

$$v_i \in \mathsf{UD}_i(K_i) \Longrightarrow v_i \in [K_i^\perp, K_i^\top].$$

Let $v_i^\perp \stackrel{\text{def}}{=} K_i^\perp$ and $v_i^\top \stackrel{\text{def}}{=} K_i^\top$, and consider any strategy $v_i \in \mathsf{UD}_i(K_i)$. If $v_i \in K_i = [v_i^\perp, v_i^\top]$ then we are done. Otherwise, suppose that $v_i < v_i^\perp$. (The other case, $v_i > v_i^\top$, can be shown analogously.)

We first claim that, for player i, reporting $v_i^\perp$ is no worse than reporting v_i. Indeed, fixing any (pure) strategy sup-profile v_{-i} for the other players and any possible true valuation $\theta_i \in K_i$, and letting $v^\perp = (v_i^\perp, v_{-i})$ and $v = (v_i, v_{-i})$, we compute that

$$\begin{aligned}
&\mathbb{E}\big[U_i\big(\theta_i, M(v^\perp)\big)\big] - \mathbb{E}\big[U_i\big(\theta_i, M(v)\big)\big] \\
&= \big(f_i(v^\perp) - f_i(v)\big)\cdot\theta_i - \big(p_i(v^\perp) - p_i(v)\big) \\
&= \big(f_i(v^\perp) - f_i(v)\big)\cdot\theta_i - \left(v_i^\perp \cdot f_i(v^\perp) - \int_0^{v_i^\perp} f_i(z, v_{-i})\, dz - v_i \cdot f_i(v) + \int_0^{v_i} f_i(z, v_{-i})\, dz\right) \\
&= \big(f_i(v^\perp) - f_i(v)\big)\cdot(\theta_i - v_i^\perp) + \int_{v_i}^{v_i^\perp} \big(f_i(z, v_{-i}) - f_i(v)\big)\, dz\ .
\end{aligned}$$

Now note that $\theta_i \in K_i$ implies that $\theta_i - v_i^\perp = \theta_i - K_i^\perp \geq 0$. Moreover, by the monotonicity of f, whenever $z \geq v_i$, it holds that $f_i(z, v_{-i}) \geq f_i(v)$. Therefore we deduce that the above difference is greater than or equal to zero. We conclude that reporting $v_i^\perp$ is no worse than reporting v_i.

Next there are two subcases. If $\mathbb{E}\big[U_i\big(\theta_i, M(v^\perp)\big)\big] - \mathbb{E}\big[U_i\big(\theta_i, M(v)\big)\big]$ equals to zero for all $\theta_i \in K_i$ and for all v_{-i}, then, using the fact that K_i has at least two distinct valuations, we conclude that for i, the allocation probability and (expected) price in outcomes $M(v_i, v_{-i})$ and $M(v_i^\perp, v_{-i})$ are the same, independent of v_{-i}. This contradicts the fact that M is weakly dominant-strategy truthful in the classical setting, since $U_i(v_i, M(v_i, v_{-i}))$ must be strictly greater than $U_i(v_i, M(v_i^\perp, v_{-i}))$ at least for some v_{-i}.

Otherwise, if there exist some θ_i^* and some v_{-i}^* that make the difference $\mathbb{E}\big[U_i\big(\theta_i, M(v^\perp)\big)\big] - \mathbb{E}\big[U_i\big(\theta_i, M(v)\big)\big]$ non-zero, it must follow that the difference is strictly positive. For such θ_i^* and v_{-i}^*, reporting $v_i^\perp$ is therefore strictly better than reporting v_i, so by definition $v_i^\perp$ weakly dominates v_i for player i, leading to a contradiction to $v_i \in \mathsf{UD}_i(K_i)$.

This concludes the proof of Theorem 1. ■

References

[AT01] Aaron Archer and Éva Tardos. Truthful mechanisms for one-parameter agents. In *Proceedings of the 2001 IEEE 42nd Annual Symposium on Foundations of Computer Science*, FOCS '01, pages 482–491. IEEE Computer Society, 2001.

[Cla71] Edward H. Clarke. Multipart pricing of public goods. *Public Choice*, 11:17–33, 1971.

[CMZ14] Alessandro Chiesa, Silvio Micali, and Zeyuan Allen Zhu. Knightian robustness of the Vickrey mechanism. *ArXiv e-prints*, abs/xxxx.xxxx, March 2014. to appear.

[MS83] Roger B Myerson and Mark A Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of economic theory*, 29(2):265–281, 1983.

[NR01] Noam Nisan and Amir Ronen. Algorithmic mechanism design. *Games and Economic Behavior*, 35:166–196, 2001.