Parametric Digital Auctions

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Abstract

We study revenue maximization for digital auctions, where there are infinitely many copies of a good for sale. There are n buyers, each of whom is interested in obtaining one copy of the good. The buyers' private valuations are drawn from a joint distribution $\boldsymbol{F}$. The seller does not know this distribution. The only information that she has are the mean μ_i and variance σ_i^2 of each buyer i's marginal distribution F_i. We call such auctions *parametric auctions*.

We construct a deterministic parametric auction that, for a wide class of distributions, guarantees a constant fraction of the optimal revenue achievable when the seller precisely knows the distribution $\boldsymbol{F}$. Furthermore, our auction is a posted price mechanism and it is *maximin optimal* among all such mechanisms. That is, it is the posted price mechanism that maximizes revenue in the worst case over an adversarial choice of the distribution.

1   Introduction

In a Bayesian auction of a digital good, there is one good for sale, available in an unlimited number of copies, and n buyers, each interested in buying one copy of the good. Each buyer i has a value v_i for the good, and the vector $(v_1, ..., v_n)$ is drawn from a joint distribution $\boldsymbol{F}$.

There are many real-world examples of such a setting, including but not limited to, selling music online, selling copies of a computer program, and giving non-exclusive licenses for a patent.

How much revenue can we guarantee to the seller in such a setting? If $\boldsymbol{F}$ is a product distribution and the seller knows $\boldsymbol{F}$, then an adaptation of Myerson's optimal auction [16] guarantees the maximum revenue that any incentive compatible mechanism can obtain.

We work with strictly weaker assumptions. Namely,

1. *$\boldsymbol{F}$ is an arbitrary joint distribution*; and

2. *The seller only knows the mean μ_i and standard deviation σ_i of each buyer i's marginal distribution F_i.*

We call such an auction a *parametric auction*. For a digital goods setting we construct a parametric auction $\mathbb{A}$, that simultaneously

(i) achieves a significant fraction of the optimal revenue, and

(ii) is *maximin optimal* among all parametric posted price mechanisms.

That is, our auction is a posted price mechanism and, for any other parametric posted price mechanism A, and for every pair of parameter vectors $(\mu_1, ..., \mu_n)$ and $(\sigma_1, ..., \sigma_n)$, there exists a distribution $\boldsymbol{F}$ —whose marginals have the given parameters— for which our $\mathbb{A}$ obtains strictly more revenue than A.

Since $\mathbb{A}$ is a *deterministic posted price mechanism*, it is dominant strategy truthful and very practical to implement. Furthermore, because we only require knowledge of the first two moments of the distribution, the seller can easily learn the information that she needs from a limited amount of data, while it may be infeasible to learn the whole distribution $\boldsymbol{F}$.

There are several auctions designed without knowledge of the full distribution. Our auction presents some advantages over these existing mechanisms. To enable a better comparison with previous work, we first establish some notation and state our results.

2   The Model

Digital Goods Settings.   We study *digital goods settings*. There is one seller, who has infinite identical copies of a good for sale. There are n buyers, each of whom wants to buy at most one copy. We assume that buyers' valuations are drawn from a joint distribution $\boldsymbol{F}$. The ith buyer's valuation V_i is distributed over some domain $D_i \subset \mathbb{R}^+$ and has marginal cumulative distribution function $F_i : D_i \to [0, 1]$, where $F_i(x) = Pr[V_i \leq x]$. We denote by

$\boldsymbol{V} = (V_1, ..., V_n)$ the vector of valuation random variables, and by $\boldsymbol{D} = D_1 \times ... \times D_n$ its domain. We denote by $\boldsymbol{v} = (v_1, ..., v_n) \in \boldsymbol{D}$ the vector of realized valuations. When we want to emphasize player i's valuation, we write the vector $\boldsymbol{v} = (v_1, ..., v_n)$ as $\boldsymbol{v} = (v_i, v_{-i})$.

Auctions. An auction is given by a pair (A, P) where $A : \boldsymbol{D} \times \Delta(\boldsymbol{D}) \rightarrow [0, 1]^n$ is an allocation rule and $P : \boldsymbol{D} \times \Delta(\boldsymbol{D}) \rightarrow (\mathbb{R}^+)^n$ is a payment rule. If the auctioneer faces a bid vector $\boldsymbol{v} = (v_1, ..., v_n)$, then she sells to player i with probability $A_i(\boldsymbol{v})$, and charges her a price $P_i(\boldsymbol{v})$.

We emphasize that $A(\boldsymbol{v}, \boldsymbol{F})$ not only depends on the valuations $v_1, ..., v_n$ but also on the distribution $\boldsymbol{F}$, but will sometimes write $A_i(\boldsymbol{v}, \boldsymbol{F}) = A_i(\boldsymbol{v})$ when $\boldsymbol{F}$ is clear from context. Furthermore, we will often take v_{-i} as fixed and write $A_i(\boldsymbol{v}, \boldsymbol{F}) = A_i(v_i)$.

We denote by $Rev((A, P), \boldsymbol{F}) = \int_{\boldsymbol{D}} \sum_{i=1}^{n} A_i(\boldsymbol{v}) P_i(\boldsymbol{v}) d\boldsymbol{F}(\boldsymbol{v})$ the expected revenue obtained by an auction (A, P) when valuations follow distribution $\boldsymbol{F}$, and by $Rev_i((A, P), \boldsymbol{F}) = \int_{\boldsymbol{D}} A_i(\boldsymbol{v}) P_i(\boldsymbol{v}) d\boldsymbol{F}(\boldsymbol{v})$ the expected revenue obtained from player i.

Truthfulness and Monotonicity. If player i obtains the good with probability A_i and pays a price P_i, her utility is $A_i v_i - P_i$. A buyer with valuation v_i can attempt to increase her utility by lying, and reporting a bid $v_i' \neq v_i$. An auction is *dominant strategy truthful (DST)* if players have no incentive to misreport their true valuation. That is, for every player i, for every valuation vector $\boldsymbol{v}$, and every $v_i' \neq v_i$, we have $v_i \cdot A_i(\boldsymbol{v}) - P_i(\boldsymbol{v}) \geq v_i \cdot A_i(\boldsymbol{v}') - P_i(\boldsymbol{v}')$, where $\boldsymbol{v}' = (v_i', v_{-i})$. It is well known [16, 1] that the auction (A, P) is DST if and only if $A_i(v_i, v_{-i})$ is monotonic in v_i and $P_i(v_i, v_{-i}) = A_i(v_i, v_{-i}) v_i - \int_0^{v_i} A_i(z_i; v_{-i}) dz_i$. An important corollary is that, for any monotonic allocation rule $A(\cdot)$, there exists a unique payment rule $P(\cdot)$ that makes the auction (A, P) truthful. Thus, it suffices to specify a monotonic allocation rule $A(\cdot)$ to specify a dominant strategy truthful auction.

Individual Rationality An auction is ex-post individually rational if $A_i(\boldsymbol{v}) \cdot v_i - P_i(\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in \boldsymbol{D}$ and all players i.

Deterministic Auctions. The value $A_i(\boldsymbol{v})$ is the probability that player i obtains the good given that the bid vector is $\boldsymbol{v}$. We focus on deterministic allocations, where $A_i(\boldsymbol{v}) \in \{0, 1\}$. If an allocation $A_i(\boldsymbol{v})$ is deterministic and truthful, then monotonicity implies that, for every v_{-i}, there exists a reserve price $p^*(v_{-i})$, possibly depending on the distribution $\boldsymbol{F}$, such that the auction sells to player i when $v_i > p^*(v_{-i})$. The payment that makes this allocation truthful is charging player i a price of $p^*(v_{-i})$ dollars if she wins.

Deterministic Posted Price Mechanisms A posted price mechanism for digital goods is a digital auction where player i is offered a take-it-or-leave-it price $p_i(\boldsymbol{F})$ that does not depend on the other players' bids. Player i gets a copy of the good if and only if $v_i > P_i(\boldsymbol{F})$. When $\boldsymbol{F}$ is a known product distribution, the optimal digital auction is a posted price mechanism. Player i is given a price $p_i^* = \text{argmax}_{p_i} p_i \cdot (1 - F_i(p_i))$.

Approximately Optimal Auctions We denote by $Rev(OPT, \boldsymbol{F})$ the optimal revenue achievable by a truthful auction which has knowledge of the distribution $\boldsymbol{F}$. For any auction A and distribution $\boldsymbol{F}$, the *competitive ratio of A* is given by

$$\frac{Rev(A, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})}.$$

Parametric Auctions Player i's valuation is a random variable V_i with mean $E[V_i] = \mu_i$, and variance $E[(V_i - \mu_i)^2] = \sigma_i^2$. We will write $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ and $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$. Informally, an auction (A, P) is *parametric* if its allocation and payment functions can be computed from the valuation vector $\boldsymbol{v}$ and the $\boldsymbol{\mu}, \boldsymbol{\sigma}$ parameters. More formally, we have the following definition.

Definition 1. *A parametric auction is a pair of functions* (A, P) *such that*

1. $A : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to [0, 1]^n$

2. $P : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

3. $A_i(\boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ *is the probability that player i wins a copy of the good when bids are $\boldsymbol{v}$ and the mean and standard deviation vectors of the distribution are $\boldsymbol{\mu}, \boldsymbol{\sigma}$.*

4. $P_i(\boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ *is the price that player i has to pay when bids are $\boldsymbol{v}$ and the mean and standard deviation vectors of the distribution are $\boldsymbol{\mu}, \boldsymbol{\sigma}$.*

For convenience of notation, whenever $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are clear we will write $A(\boldsymbol{v})$ and $P(\boldsymbol{v})$.

Monotone Hazard Rate and Regularity. Given a differentiable cumulative distribution function F_i, let $f_i(v) = \frac{d}{dv} F_i(v)$ be its induced density function. The function $h_i(v) = \frac{f_i(v)}{1 - F_i(v)}$ is called the *hazard rate* of F_i. The distribution F_i has a *monotone hazard rate* if $h_i(v)$ is increasing. The distribution F_i is called *regular* if the virtual valuation function $\phi_i(v) = v - \frac{1}{h_i(v)}$ is increasing. An immediate consequence is that any distribution with a monotone hazard rate is regular.

c-Informative Distributions. Let $c > 0$ and let F be a distribution over the real numbers with mean μ and variance σ^2. We define F to be *c-informative* if $\frac{\mu}{\sigma} > c$. We denote by $\mathcal{F}_c = \{F : \mathbb{E}[F] = \mu, Var(F) = \sigma^2, \frac{\mu}{\sigma} > c\}$ the set of all *c-informative* distributions. Note that any distribution F with a monotone hazard rate is 1-informative [1] and any distribution F over $[0, B]$ satisfying $\mu > c \cdot B$ is *c-informative*.

Note that, for every distribution F over $\mathbb{R}^+$, there exists a c such that F is *c-informative*. In particular, the sets $\mathcal{F}_c$ cover the entire space of distributions over positive valuations: $\Delta(\mathbb{R}^+) = \cup_{c>0} \mathcal{F}_c$.

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[1] Barlow, Marshall and Proschan [5] noted that for a random variable X with monotone hazard rate, $E[X^2] \leq 2E[X]^2$. Since $Var(X) = E[X^2] - E[X]^2$, this gives us $Var(X) \leq E[X]^2$. Taking square roots, we obtain $\sigma \leq \mu$.

3   Our Results

A parametric lower bound on the competitive ratio.   We construct a digital auction $\mathbb{A}$ that is competitive with the optimal auction. The competitive ratio is constant in the number of players and *does not rely on* any kind of symmetry, independence, or regularity of the distribution $\boldsymbol{F}$. However, it will be a function of the ratios $\frac{\mu_i}{\sigma_i}$. This is intuitive because if σ_i is very large compared with μ_i, then the seller has very little information about F_i, and one cannot expect her to collect significant revenue from player i.

More concretely, for any distribution $\boldsymbol{F}$ with parameter vectors $\boldsymbol{\mu}, \boldsymbol{\sigma}$ and for any player i our auction satisfies

$$\frac{Rev_i(\mathbb{A}, \boldsymbol{F})}{Rev_i(OPT, \boldsymbol{F})} \geq \rho(\frac{\mu_i}{\sigma_i})$$

where $\rho(\cdot)$ is a function of $\frac{\mu_i}{\sigma_i}$ that we define in section 5. We highlight that $\rho(\cdot)$ has the following properties:

- $\rho(\frac{\mu_i}{\sigma_i})$ is always positive when μ_i, σ_i are positive.
- When each F_i has a monotone increasing hazard rate, $\rho(\frac{\mu_i}{\sigma_i}) > 10.5\%$.

More generally, for any constant c and any c-informative distribution F with mean μ and standard deviation σ, we will have $\rho(\frac{\mu}{\sigma}) > \rho(c)$. Since for any mechanism A we have $Rev(A, \boldsymbol{F}) = \sum_i Rev_i(A, \boldsymbol{F})$, we will be able to conclude that, as long as the marginals $F_1, ..., F_n$ are c-informative distributions, our auction $\mathbb{A}$ will have a competitive ratio bounded below by a constant

$$\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \rho(c).$$

A parametric lower bound on the competitive ratio for posted price mechanisms
It is a priori possible that one could do better than a constant fraction of the optimal revenue. That is, an auctioneer who knows only the first and second moments of the distribution $\boldsymbol{F}$ could guarantee a profit arbitrarily close to optimal as the number n of players grows very large. We show in section 6 that this is impossible if we restrict the auctioneer to use posted price mechanisms. That is, for any parametric posted price mechanism A, and any parameter vectors $\boldsymbol{\mu}, \boldsymbol{\sigma}$ there exists a distribution $\boldsymbol{F}$ with the given parameters such that

$$\frac{Rev_i(A, \boldsymbol{F})}{Rev_i(OPT, \boldsymbol{F})} \leq \psi(\frac{\mu_i}{\sigma_i})$$

where $\psi(\cdot)$ is a function of $\frac{\mu_i}{\sigma_i}$ with the following properties

- $\psi(\frac{\mu_i}{\sigma_i}) < 1$
- $\lim_{\frac{\mu_i}{\sigma_i} \to \infty} \psi(\frac{\mu_i}{\sigma_i}) = 1$
- $\lim_{\frac{\mu_i}{\sigma_i} \to 0} \psi(\frac{\mu_i}{\sigma_i}) = 0$.

That is, our upper bound on the competitive ratio is a constant less than one when $\frac{\mu_i}{\sigma_i}$ is constant. When μ_i is much larger than σ_i, then the upper bound is meaningless (indeed, as μ_i becomes large, the revenue of our parametric auction $\mathbb{A}$ approaches the optimal auction).

When σ_i is much larger than μ_i, no parametric posted price mechanism can obtain any significant fraction of the optimal revenue. For these mechanisms, this result formalizes our intuition that a very large standard deviation implies an "uninformative" distribution from which we can not guarantee any revenue.

Maximin Optimality and the Digital Auction $\mathbb{A}$. As said above, our digital auction $\mathbb{A}$ is competitive with the optimal auction, where the competitive ratio depends on $\frac{\mu_i}{\sigma_i}$. Examining the competitive ratio is a meaningful way to give revenue guarantees when the full distribution is not known. However, there can be a multiplicity of parametric auctions which achieve a constant competitive ratio, and it may be difficult to decide which one is the "best." Any definition of optimality for parametric auctions needs to take into account the uncertainty that the seller has over the joint distribution $\boldsymbol{F}$. Since this is a *worst-case uncertainty*, our definition of optimality for parametric mechanisms is based on worst-case revenue maximization.

Definition 2. *Let $\mathcal{M}$ be a class of mechanisms and let $\mathcal{D}_{\boldsymbol{\mu},\boldsymbol{\sigma}}$ be the class of distributions which have parameter vectors $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$. An auction A^* is maximin optimal for $\mathcal{M}$ if $A^* \in \mathcal{M}$ and, given parameter vectors $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ we have,*

$$A^* = \underset{A \in \mathcal{M}}{\operatorname{argmax}} \quad \underset{\boldsymbol{F} \in \mathcal{D}_{\boldsymbol{\mu},\boldsymbol{\sigma}}}{\min} Rev(A, \boldsymbol{F}).$$

We highlight that, in the definition, A^* is the *unique* mechanism maximizing worst-case revenue. We show in section 7 that $\mathbb{A}$ is maximin optimal for the class of parametric posted price mechanisms.

4 Comparison with Related Work

In the spirit of the Wilson doctrine [21], a good mechanism should require as little knowledge about the valuations of the players as possible. Our paper adheres to this spirit by removing the assumption that the seller knows the distribution of buyer valuations, and replacing it by the strictly weaker assumption that the seller knows only the first and second moments of these distributions. Let us now recall prior works that, like ours, weaken the assumption that, in a Bayesian auction, the seller knows the distribution $\boldsymbol{F}$.

Baliga and Vohra [3] and Segal [19] do not make any assumptions on seller knowledge. However, their detail-free single-item auctions are competitive with the optimal auction only when the buyers' valuations are identically and independently distributed. Goldberg, Hartline, Karlin, Saks and Wright [11] consider digital auctions where the valuations are not necessarily drawn from a distribution and show how to achieve an interesting revenue benchmark. However, when the valuations are random variables, their auctions guarantee a constant fraction of the optimal auction's revenue only if these random variables are identically and independently distributed [13]. In contrast, our results apply in the more general setting where the buyers' valuations are drawn from arbitrary joint distributions.

Hartline, Mirrokni and Sundararajan [12] study how to market a digital good in a social network. To prove their main results, they show that a digital auction that sets means as reserve prices guarantees $\frac{1}{e}$ of the revenue when valuations are drawn from distributions with monotone hazard rates. When valuations are drawn from more general distributions[2], such a mechanism does not guarantee approximately optimal revenue anymore. In contrast, our mechanism guarantees good revenue for the more general class of c-informative distributions.

Dhangwatnotai, Roughgarden and Yan [9] construct auctions for the setting where the distributions $F_1, ..., F_n$ are assumed to exist, the seller does not know what they are, but has access to at least one sample from each F_i. Their auctions work for general single-dimensional matroid and downward closed environments, but require that the underlying distributions be *independent* and *regular*, assumptions that we do not make. We also emphasize that *testing* whether a distribution is indeed regular requires a large quantity of data and knowledge of the virtual valuation function. In contrast, our assumption of c-informativeness can be easily tested given only a small amount of data.

Leonardi and Roughgarden [15] consider another setting with limited seller information for digital goods. They assume that the seller knows an ordering on the buyers that is consistent with the ordering of the reserve prices that Myerson's optimal auction would charge. They construct an auction that guarantees a fraction $\frac{1}{\Omega(\log^* n)}$ of the optimal revenue. We highlight that our assumption is incomparable with theirs.

Ronen [18] and Neeman [17] give an approximately optimal auction for correlated distributions assuming that the seller knows the distribution $\boldsymbol{F}$. Dobsinzki, Fu and Kleinberg [10] improve on their results and give new auctions that have a better competitive ratio under the same assumption.

Cremer and McLean [7] show that, when the distribution $\boldsymbol{F}$ is common knowledge and sufficiently correlated, the seller can extract the full surplus from the buyers. However, their mechanism is only ex-interim individually rational, in the sense that some states of the world leave some buyers with negative utility.

Our notion of maximin optimality for digital parametric auctions follows Wald's maximin model for decision making under non-Bayesian uncertainty [20]. In this model, a decision maker has to maximize a function $f(a, s)$, that depends on her action a and an unknown state of the world s. Wald's model suggests that the player take an action $a^* = max_a min_s f(a, s)$ that maximizes the worst case payoff over all possible states of the world. As an example of this concept in mechanism design, Chung and Ely [6] study the problem of an auctioneer who knows the distribution of player valuations, but where the players can have arbitrary beliefs about each other. They show that a dominant strategy truthful auction will guarantee the maximum "worst-case revenue" in equilibrium, where the worst case is taken over the choice of players' beliefs.

A combination of approximate optimality and maximin optimality as desirable goals has been used before in mechanism design. Chassang [8] designs simple dynamic contracts for limited liability uncertain environments guaranteeing the principal a fraction of the best-first surplus, even when the distributions of returns are not i.i.d. and not ergodic. These

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[2]For example, distributions with heavy left tails.

contracts are also maximin optimal, in the respect that, under an adversarial choice of the distribution of returns, the contract maximizes the worst case revenue for the principal.

5 Digital Parametric Auctions with c-informative Distributions

We now construct a digital auction $\mathbb{A}$ which is competitive with the optimal auction. The competitive ratio will depend on the ratio $\frac{\mu_i}{\sigma_i}$. Thus, when the marginals $F_1, ..., F_n$ are c-informative, our auction will have a constant competitive ratio.

Our parametric digital auction $\mathbb{A}$

$\mathbb{A}(\boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$

   1 Find $k_i = \text{argmax}_t[(\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}]$.

   2 For each player i, set the reserve price $r_i = \mu_i - \sigma_i k_i$.

   3 Sell a copy of the good to player i if and only if $v_i > r_i$.

Theorem 1. *For any distribution $\boldsymbol{F}$ with mean vector $\boldsymbol{\mu}$ and standard deviation vector $\boldsymbol{\sigma}$, we have*

$$\frac{Rev_i(\mathbb{A}, F)}{Rev_i(OPT, F)} \geq (1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i) = (\frac{1}{2}\frac{\sigma}{\mu}k_i^3).$$

Proof.

We prove this via a series of lemmas. First, we characterize k_i in terms of $\frac{\mu_i}{\sigma_i}$.

Lemma 1. *Let $r_i = \mu_i - \sigma_i k_i$ be player i's reserve price in auction $\mathbb{A}$. We have that k_i is the unique real solution to the cubic equation $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k + k^3)$*

Proof of Lemma 1. The value k_i is obtained by maximizing the differentiable function $(\mu_i - \sigma_i k) \cdot \frac{k^2}{1+k^2}$ over $k \geq 0$. Note that finding k_i is equivalent to finding the value k maximizing $\ln(\mu_i - \sigma_i k) + 2 \ln k - \ln(1 + k^2)$. Taking derivatives of this function, we obtain that k_i satisfies the equation

$$-\frac{\sigma_i}{\mu_i - \sigma_i k_i} + \frac{2}{k_i} - \frac{2k_i}{1 + k_i^2} = 0.$$

Multiplying the denominators out, we get

$$-\sigma_i k_i(1 + k_i^2) + 2(1 + k_i^2)(\mu_i - \sigma_i k_i) - 2k_i^2(\mu_i - \sigma_i k) = 0.$$

Canceling out some terms and rearranging gives

$$-\sigma_i k_i(1 + k_i^2) + 2(\mu_i - \sigma_i k_i) = 0$$

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$$2\mu_i = 3\sigma_i k_i + \sigma_i k_i^3$$
$$\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k_i + k_i^3),$$

which is what we wanted to show.

Note that this cubic equation has exactly one real root because the function $\frac{1}{2}(3k_i+k_i^3)-\frac{\mu_i}{\sigma_i}$ is strictly increasing. Indeed, it's derivative is $\frac{3}{2} + \frac{3}{2}k_i^2 > 0$. This completes the proof of Lemma 1. $\square$

With this characterization of k_i in hand, we can give a lower bound on the expected payment from player i.

Lemma 2. *Let $\boldsymbol{F}$ be a distribution with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$. The expected revenue that auction $\mathbb{A}$ obtains from player i is at least $\mu_i - \frac{3}{2}\sigma_i k_i = \frac{1}{2}\sigma_i k_i^3$.*

Proof of Lemma 2. The expected revenue obtained from player i is $(\mu_i - \sigma_i k_i) \cdot (1 - F_i(\mu_i - \sigma_i k_i))$. However, we do not know the value $1 - F_i(\mu_i - \sigma_i k_i)$. We need to give a lower bound. To do this, we use the following one-sided version of Chebyshev's inequality.

(*Cantelli's Inequality*) For every real-valued distribution with mean μ and variance σ^2, we have
$$1 - F(\mu - \sigma k) \geq \frac{k^2}{1 + k^2}$$

From Cantelli's inequality, we obtain a bound of $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}$ on the revenue collected from player i. From Lemma 1, we know that k_i satisfies $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k_i + k_i^3)$. Multiply both sides of the equation by σ_i to obtain $\mu_i = \frac{1}{2}\sigma_i(3k_i + k_i^3)$. Now we can write $\mu_i - \sigma_i k_i = \frac{1}{2}\sigma_i(k_i + k_i^3) = \frac{1}{2}\sigma_i k_i(1 + k_i^2)$. The lower bound on the expected auction revenue becomes
$$(\mu_i - \sigma_i k_i)\frac{k_i^2}{1 + k_i^2} = \frac{1}{2}\sigma_i k_i(1 + k_i^2)\frac{k_i^2}{1 + k_i^2} = \frac{1}{2}\sigma_i k_i^3.$$

We remark that if $\sigma_i = 0$, then the player's valuation is μ_i with probability 1. Thus, the expected revenue from player i is μ_i.

Using again the fact that $\mu_i - \sigma_i k_i = \frac{1}{2}\sigma_i(k_i + k_i^3)$, we can write this revenue bound as $\mu_i - \frac{3}{2}\sigma_i k_i$. This completes the proof of Lemma 2. $\square$

We now finish the proof of Theorem 1. We have given a lower bound on the revenue that $\mathbb{A}$ obtains from each player i. We can also give an upper bound on the revenue that the optimal auction obtains from player i by noting that a truthful, individually rational auction will always charge a price p_i lower than the player's valuation v_i. Thus, the expected revenue from player i satisfies $\mathbb{E}[p_i] \leq \mathbb{E}[v_i] = \mu_i$. This upper bound holds even when valuations are correlated. Using this together with the above lemma, we can bound the player-i competitive ratio of auction $\mathbb{A}$ as follows
$$\frac{Rev_i(\mathbb{A}, \boldsymbol{F})}{Rev_i(OPT, \boldsymbol{F})} \geq \frac{\mu_i - \frac{3}{2}\sigma_i k_i}{\mu_i} = 1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i.$$

This completes the proof of the theorem. $\blacksquare$

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5.1 A constant competitive Ratio

Theorem 1 tells us that $Rev_i(\mathbb{A}, \boldsymbol{F}) \geq (1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i)Rev_i(OPT, \boldsymbol{F})$ for every player i. Note that the revenue of the auction $\mathbb{A}$ is

$$\mathbb{E}[Rev(\mathbb{A}, \boldsymbol{F})] = \mathbb{E}[\sum_{i=1}^{n} Rev_i(\mathbb{A}, \boldsymbol{F})] = \sum_{i=1}^{n}\mathbb{E}[Rev_i(\mathbb{A}, \boldsymbol{F})].$$

Thus, we can bound the revenue over all players as

$$\sum_{i=1}^{n}\mathbb{E}[Rev_i(\mathbb{A}, \boldsymbol{F})] \geq \sum_{i=1}^{n}(1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i)\mathbb{E}[Rev_i(OPT, \boldsymbol{F})] \geq \min_i(1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i)\sum_{i=1}^{n}\mathbb{E}[Rev_i(OPT, \boldsymbol{F})].$$

This gives us the competitive ratio

$$\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \min_i(1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i).$$

One key informal observation is that the competitive ratio $\min_i(1 - \frac{3}{2}\frac{\sigma_i}{\mu_i}k_i)$ is constant whenever $\frac{\mu_i}{\sigma_i}$ is bounded below by a constant for all players. Thus, for distributions with c-informative marginals, our parametric auction obtains a constant fraction of the revenue.

More formally, make the following definitions:

1. The function $k(a)$ maps a to the unique real root of the cubic equation $\frac{1}{2}(k^3 + 3k) = a$.

2. The function $\rho(a) \stackrel{\text{def}}{=} 1 - \frac{3}{2}\frac{1}{a}k(a)$. We can also write $\rho(a) = \frac{1}{2}\frac{1}{a}k^3(a)$ by our definition of $k(a)$. (See lemma 1).

We can now show the following theorem.

Theorem 2. *For any constant $c > 0$, let $\boldsymbol{F}$ be a distribution where each F_i is in the class $\mathcal{F}_c$. Then*

$$\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \rho(c).$$

Proof. Theorem 1 tells us that $\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \min_i \rho(\frac{\mu_i}{\sigma_i})$. The fact that $F_i \in \mathcal{F}_c$ implies that $\frac{\mu_i}{\sigma_i} > c$. If we show that the function $\rho(\cdot)$ is increasing, we can conclude that $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c)$ for all i, which gives us

$$\min_i \rho(\frac{\mu_i}{\sigma_i}) > \rho(c).$$

Now we show that $\rho(a)$ is an increasing function of a when $a \geq 0$. Since $\rho(a) \stackrel{\text{def}}{=} 1 - \frac{3}{2}\frac{1}{a}k(a)$, it's derivative is $\rho'(a) = \frac{3}{2}\frac{1}{a^2}k(a) - \frac{3}{2}\frac{1}{a}k'(a)$.

The function $k(a)$ was defined implicitly as $2a = k^3 + 3k$. Implicit differentiation gives us $2 = 3k^2 \cdot k'(a) + 3k'(a)$, which we can rewrite as $k'(a) = \frac{2}{3}\frac{1}{1+k^2(a)}$. Plugging this into our expression for $\rho'(a)$ we obtain

$$\rho'(a) = \frac{3}{2}\frac{1}{a^2}k(a) - \frac{1}{a}\cdot\frac{1}{1+k^2(a)}.$$

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We can multiply the above equality by $a^2 \cdot (1 + k^2(a))$ without changing the sign of $\rho'(a)$. To prove $\rho'(a) \geq 0$, it suffices to show

$$a^2 \cdot (1 + k^2(a))\rho'(a) \geq 0$$

$$\frac{3}{2}k(a)(k^2(a) + 1) - a \geq 0$$

$$k^3(a) + \frac{1}{2}k^3(a) + \frac{3}{2}k(a) - a \geq 0$$

$$k^3(a) + 0 \geq 0.$$

Since $k(0) = 0$ and k is an increasing function, we have $k^3(a) \geq 0$ when $a \geq 0$. This shows that $\rho(\cdot)$ is an increasing function when $a > 0$, and thus that $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c)$ when $F_i \in \mathcal{F}_c$. We can conclude that when all $F_1, ..., F_n \in \mathcal{F}_c$, the competitive ratio of $\mathbb{A}$ satisfies the bound

$$\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \min_i \rho(\frac{\mu_i}{\sigma_i}) > \rho(c).$$

This completes the proof of the theorem. ∎

We remark that, unlike previous auctions, we do not need the distributions $F_1, ..., F_n$ to be identical, independent, or regular. As a corollary of our theorem, we can obtain a constant competitive ratio for all distributions with a monotone hazard rate.

Corollary 1. *When each F_i has a monotone hazard rate, the competitive ratio of $\mathbb{A}$ is bounded below by*

$$\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \rho(1) > 10.5\%.$$

Proof. The fact that $\frac{Rev(\mathbb{A}, \boldsymbol{F})}{Rev(OPT, \boldsymbol{F})} \geq \rho(1)$ is immediate from Theorem 2 and the fact that a distribution with monotone hazard rate satisfies $\frac{\mu}{\sigma} > 1$. We need to show via a computation that $\rho(1) > 10.5\%$. Recall that $\rho(1) = 1 - \frac{3}{2}\frac{1}{1} \cdot k(1)$. Solving the cubic equation $k^3 + 3k = 2$ gives us $k(1) = \sqrt[3]{1 + \sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}} \cong 0.596$. Plugging this into the formula for $\rho(1)$, we obtain $\rho(1) > 10.5\%$. □

6 An upper bound on the competitive ratio of parametric posted price mechanisms

We now show an upper bound on the competitive ratio of parametric posted price mechanisms. Our upper bound depends on the ratios $\frac{\mu_i}{\sigma_i}$, and will be a constant less than 1 when these ratios are constant.

Theorem 3. *Let n be an arbitrary number of players, and let A be a parametric posted price mechanism for n players. Let $\boldsymbol{\mu}, \boldsymbol{\sigma}$ be given parameter vectors. There exists a distribution $\boldsymbol{F}$ over $(\mathbb{R}^+)^n$ with the given parameters and a function $\psi(\cdot) < 1$ (which does not depend on A, n or $\boldsymbol{F}$) such that*

$$Rev_i(A, \boldsymbol{F}) \leq \max_i \psi(\frac{\mu_i}{\sigma_i}) \cdot Rev_i(OPT, \boldsymbol{F}).$$

Proof. Since A is a posted price mechanism, player i obtains a copy of the good if she is above some reserve price $p_i = p_i(\mu_i, \sigma_i)$. If she wins the good, she pays this reserve.

Now we separate into two cases. If $p_i \geq \mu_i$, we can choose F_i to be distributed according to a random variable $C(t; \mu_i, \sigma_i)$ which takes a high value $H(t) = \mu_i + \frac{\sigma_i}{t}$ with probability $\frac{t^2}{1+t^2}$ and a low value $L(t) = \mu_i - \sigma_i t$ with probability $\frac{1}{1+t^2}$. As argued in theorem 4, this distribution has mean μ_i and variance σ_i^2. Furthermore, t is a free parameter that we can chose. Letting $t \to 0$, we have that $v_i > p_i$ with vanishing probability $\frac{t^2}{1+t^2}$. Furthermore, the revenue obtained by the auction in this case is no larger than $\frac{t^2}{1+t^2} \cdot (\mu_i + \frac{\sigma_i}{t}) = \frac{1}{1+t^2} \cdot (\mu_i t^2 + \sigma_i t)$, which goes to zero as t goes to zero.

Meanwhile, any auction with knowledge of the distribution F_i can always sell to player i at a price $q_i = \mu_i - \sigma_i t$ with probability 1. As t goes to zero, this revenue goes to μ_i. Thus, the optimal auction obtains revenue arbitrarily close to μ_i. For this case, we conclude that, for arbitrarily small $\epsilon > 0$, we can construct $F_i = C(t; \mu_i, \sigma_i)$ with sufficiently small t such that

$$\frac{Rev_i(A, \boldsymbol{F})}{Rev_i(OPT, \boldsymbol{F})} < \epsilon.$$

The second case is when $p_i < \mu_i$. Write $p_i = \mu_i - \sigma_i k$ for some $k > 0$. Now we need to choose F_i to make the revenue of our parametric auction as small as possible. Choose again $F_i = C(t; \mu_i, \sigma_i)$, but this time we will make t as large as possible. We are restricted by the fact that player i's valuation must always be a non-negative number, so $\mu_i - \sigma_i t \geq 0$. To make t as large as possible, we choose $t = \frac{\mu_i}{\sigma_i}$. Note that since $p_i = \mu_i - \sigma_i k$ must also be larger than or equal to zero, we have $k \leq t$ and $p_i \geq L(t) = \mu_i - \sigma_i t = 0$. Thus, the parametric auction A sells to player i only when $v_i = H(t)$. This happens with probability $\frac{t^2}{1+t^2}$, and the auctioneer receives a payment of p_i.

Any auctioneer with knowledge of the distribution F_i will know that v_i can only take two values: $L(t) = 0$ or $H(t) = \mu_i + \frac{\sigma_i}{t_i} = \mu_i + \frac{\sigma_i^2}{\mu_i}$. Thus, the optimal auction for this distribution will always sell at price $H(t)$, with probability $\frac{t^2}{1+t^2}$. For this case, the competitive ratio is

$$\frac{Rev_i(A, \boldsymbol{F})}{Rev_i(OPT, \boldsymbol{F})} = \frac{p_i \frac{t^2}{1+t^2}}{H(t) \frac{t^2}{1+t^2}} = \frac{p_i}{H(t)} \leq \frac{\mu_i}{\mu_i + \frac{\sigma_i^2}{\mu_i}} = \frac{\mu_i^2}{\mu_i^2 + \sigma_i^2} = \frac{(\frac{\mu_i}{\sigma_i})^2}{1 + (\frac{\mu_i}{\sigma_i})^2}.$$

Our above analysis applies only to an individual player i. Adding up over all players, we have $Rev(A, \boldsymbol{F}) = \sum_{i=1}^n Rev_i(A, \boldsymbol{F})$. We can construct the distribution $\boldsymbol{F} = F_1 \times \cdots \times F_n$, where each marginal F_i is chosen to limit the revenue that A obtains. Thus, we have

$$Rev(A, \boldsymbol{F}) = \sum_{i=1}^{n} Rev_i(A, \boldsymbol{F}) \leq \sum_{i=1}^{n} \frac{(\frac{\mu_i}{\sigma_i})^2}{1 + (\frac{\mu_i}{\sigma_i})^2} Rev_i(OPT, \boldsymbol{F}) \leq \max_i \frac{(\frac{\mu_i}{\sigma_i})^2}{1 + (\frac{\mu_i}{\sigma_i})^2} \cdot Rev(OPT, \boldsymbol{F}).$$

Thus, if we choose $\psi(\frac{\mu_i}{\sigma_i}) = \frac{(\frac{\mu_i}{\sigma_i})^2}{1 + (\frac{\mu_i}{\sigma_i})^2}$, we obtain our theorem. ∎

   Q.E.D.

   Note that the competitive ratio is a function of $\frac{\mu_i}{\sigma_i}$. An important consequence is that, as $\frac{\mu_i}{\sigma_i} \to 0$, our upper bound on the competitive ratio also goes to zero. This implies that when σ_i is much larger than μ_i, it is impossible for a parametric posted price mechanism to guarantee a good fraction of the revenue that the optimal auction would obtain.

   Another consequence is that there is a limit to the power of parametric posted price mechanisms even when $\frac{\mu_i}{\sigma_i}$ is far from zero. Our parametric posted price mechanism $\mathbb{A}$ guarantees a 10.5% fraction of the optimal revenue when $\frac{\mu_i}{\sigma_i} \leq 1$. The upper bound on the competitive ratio we just proved implies that no other mechanism can do much better. In particular, when $\frac{\mu_i}{\sigma_i} = 1$ no parametric mechanism can guarantee more than $\frac{(\frac{\mu_i}{\sigma_i})^2}{1 + (\frac{\mu_i}{\sigma_i})^2} = \frac{1}{2}$ of the optimal revenue for all distributions $\boldsymbol{F}$ with the given parameters.

7   The Maximin Optimality of Auction $\mathbb{A}$

We now prove, that no posted price mechanism can obtain more revenue than $\mathbb{A}$ in the worst case over choice of the product distribution $\boldsymbol{F}$.

Theorem 4. *The parametric auction $\mathbb{A}$ is maximin optimal among all parametric posted price mechanisms.*

Proof. In Theorem 1, we used Cantelli's inequality to show that

$$Rev(\mathbb{A}, \boldsymbol{F}) \geq \sum_{i=1}^{n} (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

To prove Theorem 4, it suffices to show that for any $\boldsymbol{\mu}, \boldsymbol{\sigma}$ and any parametric posted price mechanism $A(\boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$, there exists a distribution $\boldsymbol{F}$ with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$ such that

$$Rev(A, \boldsymbol{F}) < \sum_{i=1}^{n} (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

   Let $v_1, \ldots, v_n$ be the actual valuations of the players. Since A is a parametric posted price mechanism, it will sell to player i if and only if $v_i \geq p_i$ for some price p_i that is a function of μ_i, σ_i. Let $p_i = p_i(\mu_i, \sigma_i)$ be the reserve price that A charges to player i. Then, player i's expected payment is $p_i \cdot (1 - F_i(p_i))$.

We will prove the theorem by showing that, for each player i, there exists a distribution F_i with mean μ_i and variance σ_i on which player i's expected payment is less than $(\mu - \sigma k_i) \cdot \frac{k_i^2}{1+k_i^2}$. We split into two cases: $p_i > \mu_i$ and $p_i \le \mu_i$.

Assume $p_i > \mu_i$ and write $p_i = \mu_i + \frac{\sigma_i}{t_i}$, for some positive t_i. Consider the family of distributions $\{C(t; \mu_i, \sigma_i)\}_{t>0}$ where $C(t; \mu_i, \sigma_i)$ takes the high value $H(t) = \mu_i + \frac{\sigma_i}{t}$ with probability $\frac{t^2}{1+t^2}$, and the low value $L(t) = \mu_i - \sigma_i t$ with probability $\frac{1}{1+t^2}$. Let player i's valuation v_i be drawn from distribution $C(t; \mu_i, \sigma_i)$. When t is smaller than t_i, the auction will sell the good if and only if $v_i = H(t)$, and it will charge player i the reserve price $p_i = \mu_i + \frac{\sigma_i}{t_i}$. The expected revenue collected from player i in this case is $(\mu_i + \frac{\sigma_i}{t_i}) \cdot \frac{t^2}{1+t^2}$. Taking the limit as $t \to 0$, our expected revenue becomes arbitrarily small. In particular, it becomes smaller than $(\mu - \sigma k_i) \cdot \frac{k_i^2}{1+k_i^2}$.

Now assume $p_i \le \mu_i$. Write $p_i = \mu_i - \sigma_i t_i$. Since $A \ne \mathbb{A}$, at least one player i must have $t_i \ne k_i$. We focus on this player to obtain a strict inequality. Any player with $t_i = k_i$ will give us a weak inequality. Choose F_i to be the distribution distribution $C(t_i; \mu_i, \sigma_i)$. For this distribution, the valuation v_i will be a high value $H(t_i) = \mu_i + \frac{\sigma_i}{t_i}$ with probability $\frac{t_i^2}{1+t_i^2}$, and a low value $L(t_i) = \mu_i - \sigma_i t_i$ with probability $\frac{1}{1+t_i^2}$.

The auction A only sells to player i when $v_i > \mu_i - \sigma_i t_i$.[3] For this particular distribution, the auction will sell at price $\mu_i - \sigma_i t_i$, but only when the valuation is $H(t_i) = \mu_i + \frac{\sigma_i}{t_i}$. This happens with probability $\frac{t_i^2}{1+t_i^2}$. Thus, the expected revenue collected from player i will be

$$(\mu_i - \sigma_i t_i) \cdot \frac{t_i^2}{1+t_i^2} < (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2},$$

where the last inequality is because k_i is the unique maximum of the function $f(t) = (\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}$, and because we assumed $k_i \ne t_i$. We have shown that, when $p_i < \mu_i$, there exists a distribution F_i for which the expected revenue collected from player i is less than $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}$.

The above analysis holds for each player individually. However, since A is a digital auction, we can simply consider a product distribution $\boldsymbol{F} = (F_1, ..., F_n)$, where each F_i is chosen to limit the amount of revenue that A collects from player i. Adding up over all players, we conclude that there exists a distribution $\boldsymbol{F}$ such that

$$Rev(A, \boldsymbol{F}) < \sum_{i=1}^{n} (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}.$$

Recalling that our parametric auction $\mathbb{A}$ satisfies $Rev(\mathbb{A}, F) \ge \sum_{i=1}^{n} (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}$ for all product distributions $\boldsymbol{F}$ with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$, we obtain that $\mathbb{A}$ maximizes worst-case revenue. This concludes the proof of Theorem 4. ∎

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[3]For this proof, the strictness of the inequality matters. If we sold when $v_i \ge \mu_i - \sigma_i t_i$, then a similar argument would apply, but we would need to use a distribution $C(t_i + \epsilon; \mu_i, \sigma_i)$ for arbitrarily small values of ϵ.

8 Conclusion and Future Work

In a Bayesian setting, it is important to construct auctions that are as detail-free as possible while still guaranteeing good revenue. In this paper, we focused on a seller auctioning who has knowledge of the means and variances of players' valuations and who wants to auction a digital good. These however, are not the only parameters that can be helpful in constructing parametric auctions, or the only environments of interest. In particular, the authors, together with Daskalakis and Weinberg [2] have extended this model to use the median and other quantiles as parameters, and have constructed approximately optimal parametric auctions for matroid environments where the players' distributions are independent and regular. For downward closed environments, an analogous result applies when the distributions have a monotone hazard rate. In the future, we plan to construct auctions that use higher order moments and covariances in order to guarantee more revenue in more general settings, including auctions with multidimensional types.

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