Influence of the Projection Profile on Canny's Filter

Appendix to the paper

Capture of Hair Geometry from Multiple Images

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We show how to derive the formula used to plot the graph in Figure 6 of the paper *Capture of Hair Geometry from Multiple Images*. It demonstrates that the more extended the projection profile of an oriented filter is, the lower is the variance of its response curve – the more reliable the filter is.

We focus on the response of Canny's filter $F^{\mathrm{C}}_{(0,0)}(\theta)$ at the origin for a sinusoidal signal $s(x,y)$. For a shorter derivation and without loss of generality, we make s turn according to $-\theta$ and fix the orientation kernel filter. Since the final computation is normalized and with absolute value, we ignore the real constants and use $\sim$ to indicate proportional quantities.

Let's $G_{\sigma_x,\sigma_y}(x,y) = \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right)$ be a 2D Gaussian function with standard deviations σ_x and σ_y. Canny's kernel is then $\partial G_{\sigma_x,\sigma_y}/\partial x$ and the signal is $s_\theta(x,y) = \sin(\omega(x\cos(\theta) + y\sin(\theta)))$.

To match the sigmal wavelength and the filter pseudo-wavelength, $\sigma_x = \frac{\omega}{4}$ and we set $\sigma_y = \alpha\sigma_x$ to study the influence of α. We set $\sigma_y = \alpha\sigma_x$ and study the influence of α. With $\mathcal{F}(\cdot)$ the Fourier transform, the formula derives from:

$$F^{\mathrm{C}}_{(0,0)}(\theta) = \left| \mathcal{F}^{-1}\left(\mathcal{F}\left(\frac{\partial G_{\sigma_x,\sigma_y}}{\partial x} \right) \mathcal{F}(s_{-\theta}) \right)(0,0) \right|$$

For (u,v) the Fourier coordinates, $\delta(\cdot)$ Dirac's delta function, $\mathbf{k_0} = (\frac{\omega}{2\pi}\cos(\theta), -\frac{\omega}{2\pi}\sin(\theta))$, $\mathbf{k} = (u,v)$ and $i^2 = -1$, the classical formulæ give:

$$\mathcal{F}\left(\frac{\partial G_{\sigma_x,\sigma_y}}{\partial x} \right)(u,v) \sim iu\mathcal{F}\left(G_{\sigma_x,\sigma_y}\right) \sim iuG_{\sigma_x^{-1},\sigma_y^{-1}}(u,v)$$

$$\mathcal{F}(s_{-\theta}) \sim i\delta(\mathbf{k}+\mathbf{k_0}) - i\delta(\mathbf{k}-\mathbf{k_0}))$$

Multiplying both and using $G_{\sigma_x,\sigma_y}(-\mathbf{k}) = G_{\sigma_x,\sigma_y}(\mathbf{k})$ we get a term proportional to:

$$\cos(\theta)G_{\sigma_x^{-1},\sigma_y^{-1}}(\mathbf{k_0})\left(\delta(\mathbf{k}+\mathbf{k_0}) + \delta(\mathbf{k}-\mathbf{k_0})\right))$$

Since $\mathcal{F}^{-1}\left(\delta(\mathbf{k}+\mathbf{k_0}) + \delta(\mathbf{k}-\mathbf{k_0})\right) \sim \cos(\omega(x\cos(\theta) + y\sin(\theta)))$ that equals 1 at the origin (0,0), we have: $F^{\mathrm{C}}_{(0,0)}(\theta) \sim |\cos(\theta)G_{\sigma_x^{-1},\sigma_y^{-1}}(\mathbf{k_0})|$. With $\beta = \omega^2\sigma_x^2/8\pi^2$ and then simplifying the cosine:

$$F^{\mathrm{C}}_{(0,0)}(\theta) \sim \left|\cos(\theta)e^{\beta\left(\cos^2(\theta)+\alpha^2\sin^2(\theta)\right)}\right| = \left|\cos(\theta)e^{\beta(1-\alpha^2)\sin^2(\theta)}\right|$$

Figure 6 is a plot of $F^{\mathrm{C}}_{(0,0)}$ for $\alpha \in \{1,\cdots,10\}$ and $\beta = 1$.

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