Bidimensionality Theory and Algorithmic Graph Minor Theory

Lecture Notes for MohammadTaghi Hajiaghayi's Tutorial

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1 Introduction

Dealing with Hard Graph Problems

Many graph problems cannot be computed in polynomial time unless $P = NP$, which most computer scientists and mathematicians doubt. Examples are the Traveling Salesman Problem (TSP), vertex cover, and dominating set. TSP is to find a Hamiltonian cycle with the least weight in a complete weighted graph. A vertex cover of a graph is a subset of the vertices that covers all edges. A dominating set in a graph is a subset of the vertices such that each vertex is contained in the set or has a neighbour in the set. The decision problem is to answer the question whether there is a vertex cover resp. a dominating set with at most k elements. The optimization problem is to find a vertex cover resp. a dominating set of minimal size.

How can we solve these problems despite their computational complexity? The four main theoretical approaches to handle NP-hard problems are the following.

- Average case: Prove that an algorithm is efficient in the expected case.
- Special instances: A problem is solved efficiently for a special graph class. For example, graph isomorphism is computable in linear time on planar graphs whereas graph isomorphism in general is known to be in NP and suspected not to be in P.
- Approximation algorithms: Approximate the optimal solution within a constant factor c, or even within $1 + \epsilon$. An algorithmic scheme that gives for each $\epsilon > 0$ a polynomial-time $(1 + \epsilon)$-approximation is called a polynomial-time approximation scheme (PTAS).
- Fixed-parameter algorithms: Parametrize the problem by parameter P, which is typically the size of an optimal solution. Design an algorithm that needs $O(f(P)n^{O(1)})$ or even $O(f(P) + n^{O(1)})$ time where f is a computable function. Problems that admit such algorithms are called fixed-parameter tractable (FPT).

In the following, we deal especially with fixed-parameter algorithms and PTAS. Let us look at an example for fixed-parameter algorithms.

Example. (Fixed-parameter algorithm)
Consider the vertex cover decision problem: Given a graph G and an integer k, does a vertex cover with $\leq k$ vertices exist? This problem is known to be NP-complete, yet we want to design an algorithm that needs polynomial time in the number of vertices of G.

Can we give a simple fixed-parameter algorithm that solves the vertex cover decision problem?
We can assume that there is no vertex of degree more than k. Each vertex with more than k neighbors has to be in the vertex cover since otherwise the incident edges cannot be covered with at most k vertices.

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Moreover, we can assume that there is no matching of more than k edges because otherwise the graph has no vertex cover with at most k vertices. Thus, the total number of vertices is $\leq 2k(k-1)$, and we can try every possible vertex set of size k in $O(2^{k^2})$. Reducing a problem to a problem of size $O(f(k))$ and then solving it with brute force, as we did here, is called kernelization.

There exists a fixed-parameter tractable algorithm for vertex cover due to Chen, Kanj and Jia [2] with running time $O(kn + 1.286^k)$ which is able to solve graphs up to 400 nodes in practice.

Robertson & Seymour's Graph Minor Theory

A graph G has a graph H as a *minor* if a graph isomorphic to H can be obtained from G by a sequence of the following operations: deleting vertices, deleting edges, contracting edges. A *graph property* is a set of graphs that is closed under isomorphisms. A graph property is *minor-closed* if for each graph with that property, it holds that all its minors also have the property. Examples are planar graphs, outerplanar graphs and graphs that are embeddable on a fixed surface; but also graphs that contain a vertex cover of size at most k.

Theorem 1 (Kuratowski)**.** *A graph is planar iff it has no K_5 and no $K_{3,3}$ as a minor.*

Kuratowskis theorem characterizes planarity by two forbidden minors. This is a nice characterization since for a given graph planarity as well as non-planarity can be easily proved. Either a drawing without crossing or a forbidden minor attest the property.

Asserting Wagner's conjecture, Robertson and Seymour showed in a deep theorem that for each minor-closed graph property a result of this type holds.

Theorem 2 (Robertson & Seymour)**.** *Each minor-closed graph property can be characterized by a finite set of forbidden minors.*

Unfortunately, their result is "inherently" non-constructive, i.e. there is no algorithm that can generally determine which minors are to be excluded for a given minor-closed graph property. Moreover, the number of forbidden minors can be high: For example, for graphs embeddable on the torus more than 30,000 forbidden minors are known, yet the list is incomplete.

Robertson and Seymour devised the graph minor theory in a series of more than 20 papers, published between 1983 and 2004. In Graph Minors XIII they give for each graph H an explicit cubic algorithm that decides whether an input graph has H as a minor. Given this result and Theorem 2 the following holds.

Theorem 3 (Robertson & Seymour)**.** *Each minor-closed graph property can be decided in polynomial time (even in cubic time).*

2 Graph Classes

This section gives an overview of important graph classes used in the talk (see Figure 1).

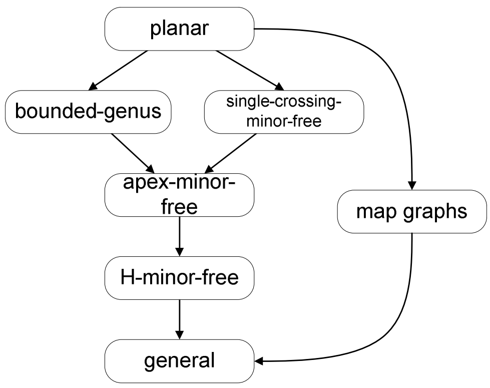

Figure 1: Graph Classes: Arrows point from more specific classes to more inclusive classes.

A graph is *planar* if and only if it can be drawn in the plane without edge intersections. The *genus* of a graph is g if it can be drawn without crossings on a surface of orientable genus g but not on a surface with orientable genus $g-1$. The *bounded-genus class* contains all graphs with fixed genus. *Map graphs* are defined similar to the dual graph g' of a planar embedding. But instead of connecting two faces (these correspond to vertices in g') with an edge in g', if they share an edge in g, we connect the faces if they share a vertex in g; also, some faces – called lakes – may be excluded from this process. These graphs can have arbitrarily large cliques. They correspond to the half-square of planar bipartite graphs.

All remaining classes in Figure 1 except the general graph class relate to excluding minors. A minor-closed graph class is H*-minor-free* for a fixed graph H if it does not contain H. An *apex graph* is a graph with a vertex v in which the removal of v results in a planar graph. A graph class is *apex-minor-free* if it excludes a fixed apex graph. A *single-crossing* graph is a minor of a graph with at most one pair of edges crossing. A minor-closed graph class is called *single-crossing-minor-free* if it excludes such a single-crossing graph.

There exist single-crossing graphs which are not drawable in the plane with at most one crossing (see Figure 2).

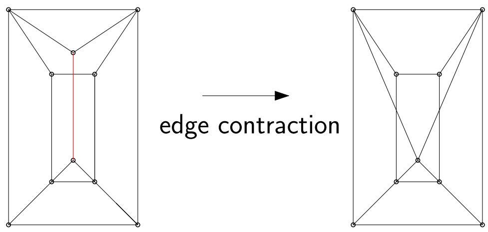

Figure 2: A single crossing that cannot be drawn with a single crossing.

In fact, a single crossing's drawing can require $\Omega(k)$ crossings if the graph has $\theta(k)$ vertices. The graph in Figure 3) presumably is a (maybe smallest possible) example for this. The first one providing a proof of this will be rewarded with a fine Australian Lager (valid until February 29, 2008).

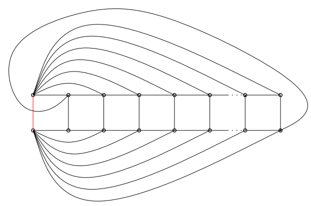

Figure 3: A graph with $2k$ vertices that can be drawn with a single crossing. But when the red edge is contracted, each drawing in the plane has at least $k-3$ crossings.

3 Structure of H-Minor-Free Graphs

An important element of Robertson and Seymour's graph minor theory and a key theorem that is used in many results of this lecture is the decomposition theorem of H-minor-free graphs [14]. It essentially says that every H-minor-free graph can be decomposed into a number of "almost-embeddable" graphs, i.e. graphs that can be embedded in a bounded-genus surface with some "extra features". This decomposition can be applied to generalize results on planar and bounded-genus graphs to H-minor-free graphs. The main operation needed in the decomposition is the notion of *clique sums* as described below.

Consider vertex-disjoint graphs G_1 and G_2 and suppose that they have cliques of size $k \geq 1$. A k*-clique sum* of G_1 and G_2 is obtained as follows: Consider a specific k-clique in each of the graphs and label their vertices with $1, \ldots, k$. Now identify the vertices with the same label to obtain a new graph with $|V(G_1)| + |V(G_2)| - k$ vertices. Finally, remove some edges from the new k-clique, if desired. Notice that this operation is *not* well-defined, i.e. there might be many ways to obtain k-clique sums of two given graphs.

The decomposition theorem of Robertson and Seymour, the so called RS-decomposition of H-minor-free graphs, says that every H-minor-free graph can be written as the clique sum of $O(1)$ graphs, where each summand is an $O(1)$-almost-embeddable graph (see the definition below) and the constants in the $O(1)$ depend only on $|V(H)|$. Demaine, Hajiaghayi, and Kawarabayashi [6] showed that such a decomposition may be found in polynomial time.

It remains to define $O(1)$-almost-embeddable graphs. A graph is $O(1)$-almost-embeddable into a bounded-genus surface if it is a bounded-genus graph with a bounded number of *vortices* and a bounded number of *apices*. A vortex is obtained by replacing a disc-like face of a bounded-genus graph with a graph of bounded pathwidth, so that the interiors of the replaced faces are disjoint. An apex is an extra vertex that may be incident to any other vertex of the bounded-genus graph (just like the apex in an apex-graph with respect to planar graphs). This completes the definitions needed for the decomposition.

4 Treewidth and Grid Minors

The treewidth of a graph is an important parameter that, roughly said, measures how far it is from being a tree. More precisely, we have the following definitions: A *tree decomposition* of a graph $G = (V, E)$ is a tree $T = (I, F)$ with subsets $x_i \subseteq V, i \in I$ as nodes (the so-called *bags*) such that

1. $\bigcup_{i \in I} x_i = V$
2. for each edge $e = \{uv\}$ exists an $i \in I$ with $u \in x_i$ and $v \in x_i$
3. for all $v \in V$ the set of nodes $\{i \in I \mid v \in x_i\}$ forms a connected subtree of T.

The *width* of a tree decomposition is the number of nodes in the largest bag of T minus 1. The *treewidth* $tw(G)$ of a graph G is the minimum width over all tree decompositions of G. More intuitively, each vertex $v \in V$ corresponds to a subtree in T, so that the subtrees of adjacent vertices overlap; the treewidth is then the maximum overlap of these subtrees minus 1. A tree decomposition with T being a path is called a *path decomposition*. The *pathwidth* $pw(G)$ of a graph G is the minimum width over all path decompositions of G.

The graph class with treewidth 1 is exactly the class of forests. The connected graphs with treewidth 2 are the series-parallel graphs. Computing the treewidth of a graph is well-motivated, since many fast algorithms for NP-hard problems exist on graphs with bounded treewidth. The typical running time of those algorithms is $2^{O(tw)} n^{O(1)}$. Unfortunately computing the treewidth itself is NP-hard, but there are constant-factor approximation algorithms, achieving a running time of $2^{O(tw)} n^{O(1)}$ for general graphs. Specifically, 1.5-approximations exist for planar graphs and single-crossing-minor-free graphs and $O(|V(H)|^2)$-approximations for H-minor-free graphs. In general graphs, there exists also a $O(\sqrt{\log \mathsf{OPT}})$-approximation computable in $n^{O(1)}$ time. Planar and bounded-genus graphs have treewidth $O(\sqrt{n})$.

Several of the defined graph classes in section 2 have powerful structural properties, as was shown in the Graph Minor Theory. A grid of size $r \times r$ is the planar graph with r^2 vertices arranged on a square grid with edges attached horizontally and vertically. Robertson, Seymour and Thomas showed in 1994 that large treewidth implies a large grid minor. A treewidth of at least 20^{2r^5} implies an $(r \times r)$-grid minor. For the class of H-minor-free graphs the following recent result shows much stronger dependence of treewidth and grid minors.

Theorem 4 (Demaine, Hajiaghayi [5])**.** *For any fixed graph H, every H-minor-free graph of treewidth w has an ($\Omega(w) \times \Omega(w)$)-grid as a minor.*

Proof sketch. The proof is based on a series of reductions using the RS-decomposition of H-minor-free graphs described in the previous section. The first observation is that the treewidth of the clique-sum of two graphs is at most the maximum of the treewidth of each one of them. So, we know that at least one summand of the RS-decomposition of the given graph G has the same treewidth as G. Removing apices and contracting vortices changes the treewidth only by a constant. The remaining bounded-genus graph is known to have a large grid minor. But there is still one important flaw in these reductions: when building clique-sums to obtain the original graph, we might *remove* edges and thus destroy the large grid-minor. To overcome this issue, Demaine and Hajiaghayi introduce a certain "approximation graph" of the almost-embeddable component; this graph is guaranteed to be a minor of G and have bounded-genus. For details, we refer to [5]. $\square$

This result is best possible up to constant factors and has several algorithmic consequences mentioned later in this lecture. For general graphs, the currently best known relation is that having treewidth more than 20^{2r^5} implies the existence of an $(r \times r)$-grid minor [15].

5 Bidimensional Graph Parameters

This section introduces bidimensionality for graph parameters. Roughly speaking, a graph parameter is bidimensional if it does not increase when performing certain operations, and it is large on specified grid-like graphs. More precisely, there are two types of bidimensionality, which we need to define: A graph parameter P is $g(r)$-*minor-bidimensional* (or just bidimensional) if it never increases under taking minors, and it is at least $g(r)$ on the $(r \times r)$-grid. The parameter P is $g(r)$-*contraction-bidimensional* if it never increases when contracting edges and it is at least $g(r)$ on *grid-like graphs*.

Which graphs exactly count as a grid-like graphs is variable, depending on the class of graphs that is considered in the context where bidimensionality is used (usually in order to prove a bound on the treewidth). For planar graphs and single-crossing-minor-free graphs, a grid-like graph is an $(r \times r)$-grid partially triangulated by additional edges that preserve planarity. For bounded-genus graphs, a grid-like graph is such a partially triangulated $(r \times r)$-grid with up to g additional edges ("handles"), where g is the bound on the genus of our graph class. For apex-minor-free graphs, a grid-like graph is an $(r \times r)$-grid augmented by additional edges such that each vertex is incident to $O(1)$ edges to nonboundary vertices of the grid. Here, $O(1)$ depends on the excluded apex graph. For more general graph classes (e.g. those excluding an arbitrary fixed minor H), contraction-bidimensionality is so far undefined.

To illustrate these definitions, let us look at some examples. A very easy graph parameter is the number of vertices of a graph, which is obviously r^2-minor-bidimensional. An easy example of a graph parameter that is contraction-decreasing but not minor-decreasing is the length of the longest cycle in a graph. This parameter is r^2-contraction-bidimensional. Likewise, the size of a dominating set (a subset of vertices such that each vertex is either in or adjacent to this subset) is contraction-bidimensional, with $g(t) = \theta(r^2)$, but not minor-decreasing. More examples of bidimensional parameters include the size of a vertex cover and the size of a maximum matching. These are both minor-bidimensional and contraction-bidimensional with $g(r) = \theta(r^2)$.

It is not always immediate to see whether a parameter is decreasing under the contraction of an edge. As an example consider the minimum size of a *clique-transversal set*, i.e., the minimum number of vertices meeting all inclusion-maximal cliques in a graph. This parameter is not contraction-decreasing, as the example in the figure shows: The four big vertices obviously form a minimum clique-transversal set of this graph. But if we know contract the red edge, a new triangle evolves in the middle of the graph, and we are forced to use a fifth vertex to cover it.

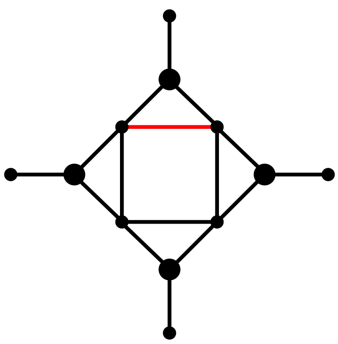

Figure 4: The four big vertices form a minimum clique transversal of this graph. After contracting the red edge, a fifth vertex is needed.

6 Obtaining FPT Results

In recent years, one of the most prominent methods to obtain FPTs and PTASs has been to bound the treewidth of the considered graph in terms of some parameter related to the problem. Such *parameter-treewidth-bounds* have been extensively used to obtain FPTs and PTASs for various problems in various graph classes in the recent time. The bidimensionality theory provides a framework for capturing many results of these kind by the following theorem:

Theorem 5 ([3, 5])**.** *For any minor-bidimensional parameter P that is at least $g(r)$ in the $(r \times r)$-grid, every H-minor-free graph G has treewidth $tw(G) = O(g^{-1}(P(G)))$. For any contraction-bidimensional parameter P that is at least $g(r)$ in an augmented $(r \times r)$-grid, every apex-minor-free graph G has treewidth $tw(G) = O(g^{-1}(P(G)))$.*

In particular, if $g(r) = \theta(r^2)$, then these bounds become $\mathrm{tw}(G) = O(\sqrt{P(G)})$. But as we mentioned earlier, many of the considered problems can be solved on graphs of bounded treewidth in time $2^{O(\mathrm{tw}(G))}n^{O(1)}$. This implies that all these problems admit *subexponential* fixed parameter algorithms with running time $2^{O(\sqrt{k})}n^{O(1)}$ on the mentioned graph classes, where k is the bidimensional parameter, typically the solution size. Examples of these problems include vertex cover, minimum maximal matching, dominating set and unweighted TSP tour.

It is important to note that for contraction-bidimensional parameters, these results are limited to apex-minor-free graphs. This is due to the fact that the so called diameter-treewidth-property – one of the first and most important proven parameter-treewidth-bounds obtained by Eppstein [10] – holds exactly for this class of graphs. Still, this does not imply that problems that are contraction-closed but not minor-closed do not admit FPTs beyond this class: the dominating set problem has been shown to admit a $2^{O(\sqrt{k})}n^{O(1)}$-time FPT on H-minor-free graphs, map graphs and in fact, all fixed powers of H-minor-free graphs, whereas it is contraction-closed but not minor-closed. So, there is still hope to obtain these kinds of results for other parameters of this kind – maybe even derive a theory of graph contractions similar to the theory of graph minors.

7 Obtaining PTASs

Since Lipton and Tarjan's [13] separator theorem for planar graphs in 1979, several PTASs for optimization problems on planar graphs and their generalizations have been devised. They were either based on this or consequent separator theorems or on another seminal framework introduced by Baker [1] in 1994 using layerwise decomposition. The bidimensionality theory captures many of these results and generalizes each of these methods in the following way:

The separator approach is based on finding small separators in the input graph, solving the problem on the resulting smaller graphs recursively, and merging the computed solutions. The size of the separator plays an important role in this process and has been usually bounded in terms of the size of the input graph. Using the parameter-treewidth bound, one can find small separators in terms of the *solution size* and this boosts the power of this approach by much. The idea is to find a tree decomposition with treewidth bounded in the size of the parameter and choose the most balanced cut that it provides. Since the treewidth is bound by the size of the parameter, so is the size of the derived cut. Demaine and Hajiaghayi obtain the following result:

Theorem 6 ([4, 5])**.** *Consider a $\theta(r^2)$-minor-bidimensional problem that satisfies a certain separation property described below and that can be solved in time $h(tw(G))n^{O(1)}$. Then the problem admits a PTAS with running time $h(O(1/\epsilon))n^{O(1)}$ on all H-minor-free graphs. The same results holds for $\theta(r^2)$-contraction-bidimensional problems on apex-minor-free graphs.*

The required separation property is somewhat technical and differs slightly for minor-bidimensional and contraction-bidimensional parameters but is roughly as follows:

- The solution on disconnected graphs is the union of solutions of each connected component.
- Given a solution to $G - C$, one can compute a solution to G at an additional cost of $\pm O(|C|)$.
- A solution S of G induces on a connected component X of $G - C$ a solution with size $|S \cap X| \pm O(|C|)$.

This results in PTASs in H-minor free graphs for vertex cover, face cover, minimum maximal matching and feedback vertex set, among others. On apex-minor-free graphs one obtains PTASs for various kinds of dominating set problems.

Baker's layerwise decomposition, the second approach for designing PTASs mentioned above, has been generalized by Eppstein [10] and Grohe [12] through the notion of *bounded local treewidth*. A graph is said to have bounded local treewidth if for each connected subgraph with bounded diameter, the treewidth of the subgraph is bounded. The relationship between the diameter and the bound on the treewidth is crucial in the running time of the obtained PTASs and this bound could be substantially improved using Theorems 4 and 5 and the bidimensionality of diameter [5]: the running time has been improved from $2^{2^{2^{O(1/\epsilon)}}} n^{O(1)}$ to $2^{O(1/\epsilon)} n^{O(1)}$. Frick and Grohe [11] also showed that any graph property expressible in first-order-logic, can be decided in linear time on graphs with bounded local treewidth. Using Theorem 4, one can improve the running time of the obtained PTASs from $2^{2^{2^{O(1/\epsilon)}}} n^{O(1)}$ to $2^{2^{O(1/\epsilon)}} n^{O(1)}$. Also, by using some more sophisticated techniques, a PTAS could be obtained for connected dominating set on apex-minor-free graphs, which was previously unknown [4].

Another generalization of Baker's approach has been achieved by the following theorem:

Theorem 7 ([6]). *The edge set (vertex set) of any H-minor-free graph can be partitioned into k graphs such that the union of any $k-1$ of them has bounded treewidth.*

This gives rise to PTASs for H-minor-free graphs for several NP-complete problems, such as minimum color sum, maximum P-matching and max-cut.

Finally, in a very recent paper, the following result has been obtained:

Theorem 8 ([7]). *The edge set of any bounded-genus graph can be partitioned into k graphs such that* contraction *of each one results in a bounded treewidth graph.*

This theorem results in PTAS for *weighted* TSP and minimum 2-connected subgraph on bounded-genus graphs. Whether this theorem is true for all H-minor-free graphs is still open.

8 Recent Improvements

Recently, a series of results concerning the design of subexponential parameterized algorithms on NP-hard graph problems was presented by Dorn, Fomin and Thilikos (see [9] and [8]). They showed that problems like *k-Longest Path* are bidimensional in order to use the Bidimensionality Theory to get a branch- or tree decomposition whose width is bounded by a function of sublinear size of the parameter k, here $O(\sqrt{k})$.

In general, there are many problems that can be tracked with dynamic programming on these decompositions as k is fixed. Dorn, Fomin and Thilikos focused on improving the running time of these algorithms to a subexponential time in k, typically $f(2^{\sqrt{k}}) \cdot n^{O(1)}$, by bounding the steps of dynamic programming with the Catalan numbers for different input graph classes. Among these are the classes of planar graphs, graphs with bounded genus and H-minor-free graphs ([9]). Furthermore they proposed a way to get an exponential speed-up for a class of problems by using specific matrices for saving and comparing the solutions in the dynamic program.

9 Open Problems

In this last section, we present a list of interesting questions and conjectures and directions for future research on the topic of this lecture:

- Is it possible to extend the bidimensionality theory beyond H-minor-free graphs, maybe even apply it to general graphs? Robertson, Seymour and Thomas showed in 1994 that even in general graphs large treewidth implies a large grid minor. But the relationship is still too loose: a treewidth of more than 20^{2r^5} implies the existence of an $(r \times r)$-grid minor. The only known lower bound is that some graphs of treewidth $\Omega(r^2 \log r)$ have no grid larger than $O(r) \times O(r)$. There is a huge gap in between these bounds and it is conjectured that a treewidth of $r^{\theta(1)}$, maybe even r^3 could suffice to certify a grid-minor of size $(r \times r)$. If this is true, it would be a huge step towards generalizing many of the results mentioned in this lecture.

- Is there a theory of graph contractions for handling contraction-closed properties? Currently, contraction-bidimensionality can not be extended beyond apex-minor-free graphs to still obtain parameter-treewidth bounds as is shown by considering the dominating set problem. Also, an analog of Wagner's conjecture for contractions of graphs is not true in general: There is an infinite sequence of connected graphs such that none can be obtained from another via edge-contractions, see Figure 5.

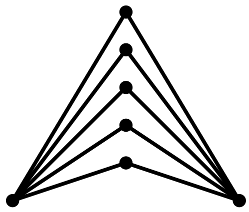

Figure 5: Every graph in the sequence $G_i = K_{2,i+2}$ is triangle-free, but contains 4-cycles. After contraction of an arbitrary edge, the graph will contain a triangle which we cannot get rid of by further contractions while preserving the 4-cycles.

Still, the dominating set problem admits an FPT on all fixed powers of H-minor-free graphs, so there might be some hope in generalizing this result to other classes of problems. One possible consideration could be to replace parameter-treewidth bounds with the notion of parameter-cliquewidth bounds.

- Is it possible to generalize the PTASs beyond bidimensional parameters, e.g. parameters that involve vertex- or edge-weights or for subset-type problems like Steiner tree? In the weighted case, the notion of being large on a grid is no longer well-defined and depends on the weights of vertices and edges in the chosen grid. In the subset-type problems, the size of the solution depends on the subset of vertices that are to be included in the solution and so, also in this case, being large on a grid is not well-defined. But very recently, a PTAS for Steiner tree on planar graphs has been discovered, so there might be some unifying method to also capture these kinds of problems.
- Finally, a crucial open problem is the reduction of the constant factors hidden in almost all algorithms on H-minor-free graphs. These constants are so extremely large in terms of the size of the excluded minor H, that it makes these algorithms absolutely impractical. A known lower bound is $\Omega(\sqrt{|V(H)|}\log|V(H)|)$ – which is very small. It is conjectured that these constants can be reduced to $|V(H)|^{O(1)}$ or maybe even to $O(|V(H)|)$. If true, this would be a substantial speed-up in pushing these algorithms towards practicality.

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