Supplementary Material for “A Principled Deep Random Field Model for Image Segmentation”

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In this supplement, we provide details on the multi-label model and also prove some of the theoretical results in the main paper.

1. Multi-label models

Let $\mathcal{L}$ be the set of all labels that a node can take. We will denote labels $a \in \mathcal{L}$ by fractional characters.

The multi-label extension of the directed cooperative cut energy that is defined in the main paper is

$$
\Psi_g(x) = \sum_{a \in \mathcal{L}} F_g(\sum_{(i,j) \in g} \psi_a(x_i, x_j)),
$$

(1)

where the pairwise function $\psi_a$ is defined as (see also [3, Sec. 5.4.1])

$$
\psi_a(x_i, x_j) = \begin{cases} 
\theta_{ij} & \text{if } x_i = a \text{ and } x_j \neq a \\
0 & \text{otherwise}. 
\end{cases}
$$

(2)

As before, we introduce group variables $h_{g,a}$, indexed by edge groups and labels, because we have one function $F_g$ for each label. If $F_g(a) = \min\{a, T\}$ is the truncation with one breakpoint, then we can write

$$
\Psi_{g,a}(x) = \min_{h_{g,a} \in \{0,1\}} h_{g,a} \left( \sum_{(i,j) \in g} \psi_a(x_i, x_j) \right) + (1 - h_{g,a}) T.
$$

(3)

If the vector $h$ of variables $h_{g,a}$ is fixed, then the entire energy becomes a sum of pairwise potentials of the form

$$
\psi_{i,j}(x_i, x_j) = \begin{cases} 
0 & \text{if } x_i = x_j \\
\theta_{ij}(a) + \theta_{ji}(b) & \text{if } x_i = a, x_j = b, 
\end{cases}
$$

(4)

where $\theta_{ij}(a)$ is the weight of $\psi_a(x_i, x_j)$ under the current assignment of $h_{g,a}$. In the above case, $\theta_{ij}(a) = \theta_{ij} h_{g,a}$.

We will perform expansion moves [1] with such a potential. For a given assignment $x \in \mathcal{L}^n$, an expansion move with respect to label $a$ is allowed to change any label of $x$ to $a$, but can make no other changes. Formally, define the set

$$
\mathcal{X}(x, a) = \{ y \in \mathcal{L}^n | y_i = x_i \text{ or } y_i = a \text{ for all } 1 \leq i \leq n \}.
$$

(5)

An expansion move with respect to a label $a$ and the current labeling $x$ finds

$$
y' \in \arg \min_{y \in \mathcal{X}(x, a)} E(y).
$$

(6)

A stationary point $x'$ with respect to all labels is one that for all labels $a \in \mathcal{L}$ satisfies $x' \in \arg \min_{y \in \mathcal{X}(x, a)} E(y)$.

For pairwise potentials such as 4, expansion moves can be computed as the minimum cut in a graph. This proves Lemma 2 in the main paper:

**Lemma 2.** The multi-label model can be reduced to a non-submodular pairwise model analogous to the binary model. If $|\mathcal{L}|$ and $|\mathcal{G}|$ are constants, then, with the help of $|\mathcal{L}||\mathcal{G}|$ auxiliary variables, we can compute an exact expansion move in polynomial time.

In the sequel, we will denote the vector of all $h_{g,a}$ variables by $h$. We re-state Theorem 1 in the main paper in a bit more detail:

**Theorem 1.** Let $x^* \in \arg \min E(x)$ be an optimal MAP labeling for a cooperative cut energy composed of terms of the form (1). For a given assignment $h \in \{0,1\}^{|\mathcal{G}||\mathcal{L}|}$ of the group variables, let $x(h)$ be a stationary point of the expansion moves with respect to all labels. Then

$$
\min_{h \in \{0,1\}^{|\mathcal{G}|}} E(x(h)) \leq 2c E(x^*),
$$

(7)

where $c = \max_{a, b \in \mathcal{L}, g \in \mathcal{G}} F_g'(0)/F_g'(\sum_{(i,j) \in g} \theta_{ij})$ is the ratio of the largest and smallest slopes of $F$.

**Proof.** For each $h$, we find a labeling $x(h)$ that is a stationary point with respect to all labels. In the end, we will take the best of the solutions $x(h)$ that we found. This can be done via a variant of the graph cut algorithm in [1]. To ease
notation in the proof, we introduce the notation $E_h(x, h)$ for the energy function that is a function of $h$ (instead of minimizing over it as in Equation 3. With this notation, $E(x) = \min_h E_h(x, h)$.

An adaptation of Theorem 6.6 in [1] implies that

$$E_h(x(h), h) \leq 2\epsilon \min_{y \in \mathbb{C}^n} E_h(y, h). \quad (8)$$

The constant $\epsilon$ arises since the slopes of $F$ (determined by the assignment of $h$) scale the pairwise weights $\theta_{ij}$, and this leads to label-sensitive pairwise potentials in the framework of [1]. Let $h^*$ be the optimal assignment of $h$ for the optimal solution $x^*$, i.e., $E_h(x^*, h^*) = E(x^*)$. Since the bound (8) holds for all assignments $h$, we get that

$$\min_{h \in \{0, 1\}^{|G|}} E(x(h)) = \min_{h \in \{0, 1\}^{|G|}} \min_{h' \in \{0, 1\}^{|G|}} E_h(x(h), h') \leq \min_{h \in \{0, 1\}^{|G|}} E_h(x(h), h) \leq E_h(x(h^*), h^*) \leq 2\epsilon \min_{y \in \mathbb{C}^n} E_h(y, h^*) = 2\epsilon E_h(x^*). \quad (9)$$

2. Arbitrary monotone concave functions

We consider the energy

$$E(x) = \sum_i \psi_i(x_i) + \sum_{g \in G'} \Psi_g(x), \quad \text{where} \quad (14)$$

$$\Psi_g(x) = F_g(\sum_{i,j \in E_g} \psi_{ij}(x_i, x_j)). \quad (15)$$

We make the following assumptions:

1. the pairwise potentials are of the form $\psi_{ij}(x_i, x_j) = \theta_{ij}|x_i - x_j|_+ \geq 0$ or $\psi_{ij}(x_i, x_j) = \theta_{ij}|x_i - x_j| \geq 0$;
2. the functions $F_g: \mathbb{R}_+ \to \mathbb{R}_+$ are nonnegative, monotone increasing scalar concave functions that satisfy $F_g(\lambda y) \leq \lambda F_g(y)$ for all $y \geq 0$;
3. the energy $E$ is nonnegative.
4. $|G'|$ is constant.

We re-state Lemma 1 from the main paper:

**Lemma 1.** If the energy (14) satisfies (1)-(4), then there is an FPTAS for minimizing this energy, i.e., there is an algorithm that runs in time polynomial in $1/\epsilon$ and $n$ and returns a solution $x$ with $E(x) \leq (1 + \epsilon)E(x^*)$, where $x^*$ is the optimizing MAP assignment.

**Proof.** We will treat the unary potentials as an additional edge group (this is the group of terminal edges), and set $k = |G| + 1$. Let $M$ be such that the energy functions takes values between $1/M$ and $M$. We create a set of slopes $A = \{\alpha = rw \mid r \in \mathbb{R}, w \in W\}$, where $R = \{2^0, 2^1, \ldots, 2^{|\log_2 M|}\}$ and $W = \{1, 2, \ldots, \lfloor \frac{2(k-1)}{\epsilon} \rfloor\}$. We will essentially represent $\Psi_g$ by a piecewise linear function with pieces

$$\hat{\Psi}_g(x; \alpha) = \sum_{(i,j) \in g} \alpha \psi_{ij}(x_i, x_j) \quad (16)$$

with slopes $\alpha \in A$.

To see how such a function connects to an approximation by functions $\Psi'(x) = \min\{\beta \sum_{(i,j) \in g} \psi_{ij}(x_i, x_j), T\}$, observe that fixing $h$ in the algorithm in the main paper corresponds to assigning a “slope” $\sum \beta_i h_t$ to each edge group. We find a minimizer for each slope, and, among those minimizers, select the one minimizing the actual energy. We will see that doing the same for the slopes $\alpha$ will suffice. The pieces (16) can be written in terms of the functions $\hat{\Psi}(x) = \min\{\beta \sum_{(i,j) \in g} \psi_{ij}(x_i, x_j), T\}$. To do so, we sort the slopes in $A$ in increasing order, and number them $\alpha_1, \alpha_2, \ldots$. The corresponding $\beta_i$ are then $\beta_1 = \alpha_1$ and $\beta_i = \alpha_i - \alpha_{i-1} (i > 1)$, so that $\alpha_i = \sum_{i<j} \beta_i$. (Those $\beta_i$ are only needed for the conceptual connection.)

Recall that a particular assignment $h$ in the algorithm in the main paper corresponds to assigning a slope $\alpha_g \in A$ to each group. We hence imitate the algorithm by computing the minimizers $x(a)$ for all $a \in A^G$ and for

$$\hat{E}(x; a) = \sum_{g \in G'} \hat{\Psi}_g(x; \alpha_g), \quad (17)$$

where $G'$ is the extended set of edge groups that includes the extra group for the unary potentials. We then evaluate the energy $E$ at each assignment $x(a)$, and choose the best among those assignments.

To analyze this strategy, we will take the cut viewpoint and draw connections to multi-objective optimization to be able to use ideas from [5]. For the cut viewpoint, we introduce binary variables $y_{ij} = |x_i - x_j|$. Each assignment $x$ corresponds to a cut $y$ and vice versa.

We can write the energy as

$$E(x) = \sum_{g \in G'} \sum_{(i,j) \in E_g} \theta_{ij}^g y_{ij}, \quad (18)$$

where $\theta_{ij}^g = 0$ if $(i,j) \notin G'$. Written in this form, the energy can be viewed as a function combining $k$ linear objectives $g(y) = \sum_{ij} \theta_{ij}^g y_{ij}$ into one objective by using a concave function $F(\theta(1)y, \ldots, \theta(k)y) = \sum_g F_g(\theta(1)y)$. A theorem in [2] says that the set $\{x(a) \mid a \in A^G\}$ is an $\epsilon$-approximate convex Pareto-optimal front corresponding to the $k$ linear functions $\theta(1)y$ for the constraint that
is the indicator vector of a cut. An approximate convex Pareto-optimal set \( C_\epsilon \) is a set of solutions such that for each feasible \( y \in [0,1]^2 \) there exists an \( y' \in C_\epsilon \) such that \( \theta(g)y' \leq \theta(g)y \) for all \( g \in G' \).

Lemma 3.2 in [5] states that the convex hull of \( C_\epsilon \) contains a \((1 + \epsilon)\)-optimal point. Since our energy is concave, the minimum over the convex hull is attained at a corner point, and therefore the search over all corner points (i.e. over the set \( C_\epsilon \)) will yield a \((1 + \epsilon)\)-approximate solution.

Finally, the cardinality \(|A|\) (which determines the number of optimization problems to solve) is polynomial in \( 1/\epsilon \) and \( n \).

### 3. Details on experiments

Here, we provide some more details on how the potentials in the experiments were computed.

The unary potentials are computed by fitting a Gaussian mixture model with 5 components to pixels of seed regions. We added user scribbles to the MSRC data for the multilabel experiments to be compatible with the binary label experiments.

We use an 8-neighbor graph structure and contrast-dependent Potts pairwise potentials \( \theta_{ij} = 2.5 + 47.5 \exp(-0.5\|I_i - I_j\|^2/\sigma) \), where \( \sigma \) is the mean of the color gradients in the image.

As in [4], the edge groups were defined by defining a 3-dimensional feature vector \( \phi(i,j) = I_i - I_j \) for each edge \((i,j)\) using the pixel RGB values \( I_i \). We then cluster the edges using these features (using \( k \)-means), and each cluster becomes a group \( g \in \mathcal{G} \). The discount functions \( F \) were the same in the binary and multi-label case:

\[
\Psi_g(x) = \lambda \min \left\{ \sum_{ij \in E_g} \theta_{ij}(x_i - x_j)_+, \sum_{ij \in E_g} \alpha \theta_{ij}(x_i - x_j)_+ + \theta_g \right\}
\]

For the multi-label functions, we adapt this function according to Section 1 in this supplement and make it label-dependent.

### References


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\(^1\)The “feasible set” here is the convex hull of all cut indicator vectors