Discrete Mathematics

Combinatorics

Counting principle: If a choice consists of k steps, of which the first can be made in n_1 ways, the second in n_2 ways, … , and the k^{th} in n_k ways, then the whole choice can be made in $n_1 n_2 \ldots n_k$ ways.

Factorials: $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$

Permutations: A permutation is an arrangement of objects where order matters. (123 and 213 are considered different permutations of the digits 1, 2, and 3).

${}_nP_r$ is the number of permutations of r objects chosen from n objects.

$${}_nP_r = \frac{n!}{(n-r)!}$$

Special cases: there are $n!$ ways of arranging all n objects.

Repeated objects: In an arrangement of n objects, if there are r_1 objects of type 1, r_2 objects of type 2, … r_k objects of type k, where objects of the same type are indistinguishable, then there are $\frac{n!}{r_1!r_2!\cdots r_k!}$ ways to arrange the n objects.

Circular Permutations: If n objects are arranged in a circle, there are $(n-1)!$ possible arrangements.

"Key-ring" permutations: If n objects are arranged on a key ring, there are $\frac{(n-1)!}{2}$ possible arrangements.

Combinations: In a combination, the order of objects does not matter (123 is the same as 213).

${}_nC_r$ is the number of combinations of r objects chosen from n objects.

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Sets:

For sets *A* and *B*,

Union: $A \bigcup B$ is the set that contains the elements in either *A*, *B*, or both.
Intersection: $A \bigcap B$ is the set that contains only elements that are in both *A* and *B*.
Complement: A' is the set of all elements not in *A*.
Inclusion-Exclusion principle: If *n(S)* is the number of elements in set *S*, then
$$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B).$$

This can be extended for more than two sets. (ex. For sets *A*, *B*, and *C*, $n(A \bigcup B \bigcup C) = n(A) + n(B) + n(C) - n(A \bigcap B) - n(B \bigcap C) - n(A \bigcap C) + n(A \bigcap B \bigcap C)$.

Probability:

If an experiment can occur in exactly *n* ways, and if *m* of these correspond to an event *E*, then the probability of *E* is given by
$$P(E) = \frac{m}{n}$$
P(A and *B)* = $P(A \cap B) = P(A)P(B)$ if A and B are independent events.
P(A or *B)* = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability: the conditional probability of an event *E*, given an event *F*, is denoted by *P(E/F)* and is defined as $P(E / F) = \dfrac{P(E \cap F)}{P(F)}$.

Pigeonhole principle: If there are more than *k* times as many pigeons as pigeonholes, then some pigeonhole must contain at least *k+1* pigeons. Or, if there are *m* pigeons and *n* pigeonholes, then at least one pigeonhole contains at least $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Ex. Consider any five points P_1, P_2, P_3, P_4, and P_5 in the interior of a square *S* with side length *1*. Denote by d_{ij} the distance between points P_I and P_j. Prove that at least one of the distances between these points is less than $\frac{\sqrt{2}}{2}$.

Solution: Divide *S* into four congruent squares. By the pigeonhole principle, two points belong to one of these squares (a point on the boundary can be claimed by both squares). The distance between these points is less than $\frac{\sqrt{2}}{2}$. (Problem and solution from Larson, number 2.6.2).