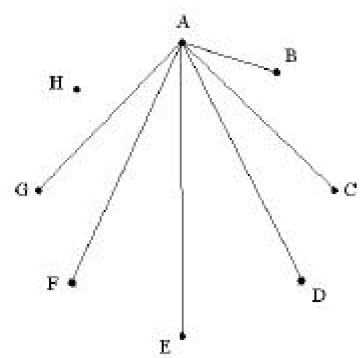
Strategies

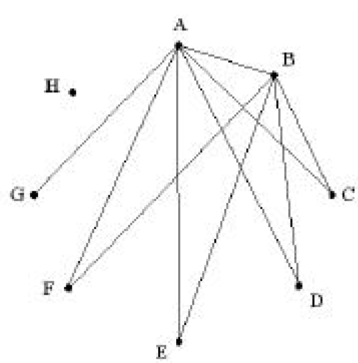
1. **Search for a pattern:**
   Ex. Compute $\sqrt{(31)(30)(29)(28)+1}$ (no calculators) (1989 AIME, #1)

   Starting at 1 instead of 28, we see that
   $$\sqrt{(3)(2)(1)(0)+1} = \sqrt{1} = 1 = 1^2 + 0$$
   $$\sqrt{(4)(3)(2)(1)+1} = \sqrt{25} = 5 = 2^2 + 1$$
   $$\sqrt{(5)(4)(3)(2)+1} = \sqrt{121} = 11 = 3^2 + 2$$
   $$\sqrt{(6)(5)(4)(3)+1} = \sqrt{361} = 19 = 4^2 + 3$$
   Then it appears $\sqrt{(n+3)(n+2)(n+1)n+1} = (n+1)^2 + n$, so the solution to our problem would be $29^2 + 28 = 869$. Multiplying out the polynomials will show that the formula is accurate.

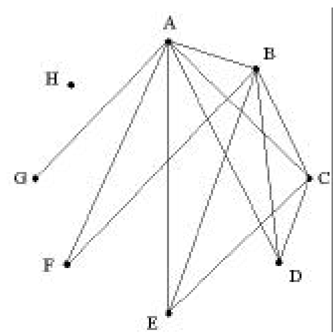
2. **Draw a Figure:**
   Ex. Mr. and Mrs. Adams recently attended a party at which there were three other couples. Various handshakes took place. No one shook hands with his/her own spouse, no one shook hands with the same person twice, and no one shook his/her own hand. After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Adams shake? (Larson 1.2.4)

   A diagram with dots representing people is helpful. The numbers that Mr. Adams received must have been 0, 1, 2, 3, 4, 5, and 6. Suppose A shook hands with 6 other people (B through G, for example). This is represented by the diagram to the right. H must be the person who shook 0 hands, and A and H must be a couple since A shook hands with everyone else. Now suppose B shook 5 hands (A, C, D, E, and F, for example). This diagram is shown below.





   G must be the person who shook 1 hand, and B and G must be spouses. If C is the person who shook 4 hands, we find similarly that F shook 2 hands. Completing the diagram, we see that D and E both shook 3 hands. They must be Mr. and Mrs. Adams.



3. **Formulate an Equivalent Problem:**
Sometimes when the problem or the calculations are complicated, the problem can often be rewritten or manipulated into a different form that is easier to solve. Ways to do this include algebraic or trig manipulation or substitution, use of a one-to-one correspondence, or reinterpreting the problem into a different subject.

Ex. On a circle *n* points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent, (except at the endpoints), how many points of intersection are there? (Larson 1.3.5)

When four points are selected, connecting all the points together produces a quadrilateral with two intersecting diagonals. Therefore with any selection of 4 points, there is exactly one point of intersection. The problem is equivalent to the number of ways to chose 4 points from *n* points, which is just $\binom{n}{4}$.

4. **Modify the Problem:**
This method is closely related to number 3. It is very general, and many types of problems could potentially fall under this category.

5. **Choose Effective Notation:**
Problems can often be simpler depending on the notation used.
Ex. The sum of 5 consecutive terms is 195. Of these terms, what is the largest one given a common difference of 13.

If $a = 13$, one might call the largest term x and the other terms $x - a$, $x - 2a$, $x - 3a$, and $x - 4a$. However, letting x be the middle term produces the other terms $x - 2a$, $x - a$, $x + a$, and $x + 2a$. The a's cancel out nicely when added together, so $5x = 195$, or x is 39. Then the largest term is $39 + 26$ or 65.

6. **Exploit Symmetry:**
Using symmetry in certain problems often reduces the amount of work that must be done. For example, when multiplying out a polynomial such as $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$, all the variables can be interchanged, so if there is an a^3 term, there must be a b^3 and a c^3 term with the same coefficient. The terms $a^2b, a^2c, ab^2, b^2c, ac^2, bc^2$ will all have the same coefficients as well. Also, in another example, when graphing a function like $|x|+|y|=4$, there is symmetry across both axes, so only one quadrant must be plotted before reflecting across the axes.

7. **Divide into Cases:**
Some problems can be divided into smaller sub-problems that can be solved individually.
Ex. When finding the probability of getting at least 7 heads if a coin is flipped 10 times, the problem is usually split into finding the probability of exactly 7, 8, 9 and exactly 10 heads.

8. **Work Backwards**
Many proofs and some problems are easiest if worked backwards and then reversing the steps to obtain the desired result.
Ex. Prove that the arithmetic mean of a number is always greater than or equal to the geometric mean.

Suppose that this is true. Then $\frac{x+y}{2} \geq \sqrt{xy}$. Squaring, we have $x^2 + 2xy + y^2 \geq 4xy$

This simplifies to $(x-y)^2 \geq 0$, which is obviously always true. We can reverse our steps so the proof is valid.

9. **Argue by Contradiction:**
Some proofs are done by assuming the opposite of what you want is true, and then working until a contradiction is reached.

Ex. Prove the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}$ diverges. (Larson 1.9)

Suppose the series converges, and the sum is r. Then

$$r = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\cdots+\frac{1}{n-1}+\frac{1}{n}$$

$$r > \frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\frac{1}{8}+\frac{1}{8}+\cdots\frac{1}{n}+\frac{1}{n}$$

$$r > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

But this implies that $r > r$, which is a contradiction. Therefore the series must diverge.

10. **Pursue Parity**
Whether a number is even or odd can help solve problems that otherwise seem unrelated.
Ex. Place a knight on each square of a 7-by-7 chess board. Is it possible for each knight to simultaneously make a legal move? (Larson 1.10.2)

Assume a chessboard is colored in the usual checkered pattern. The board has 49 squares; suppose 24 of them are white and 25 are black. Consider 25 knights which rest on the black squares. If they were to make a legal move, they must move onto 25 white squares. But this is impossible, since there are only 24 white squares.

11. **Consider Extreme Cases:**
Often if a problem says that something works for all cases, it must work in specialized cases. For example, a theorem that works for all triangles must work for equilateral or right triangles. Testing extreme cases can either provide counterexamples or help to determine a pattern for general cases.

12. **Generalize:**
Sometimes a more general case is easier to solve than a specific case. Replacing a specific number with a variable may make a solution more visible.