LECTURE 7

LECTURE OUTLINE

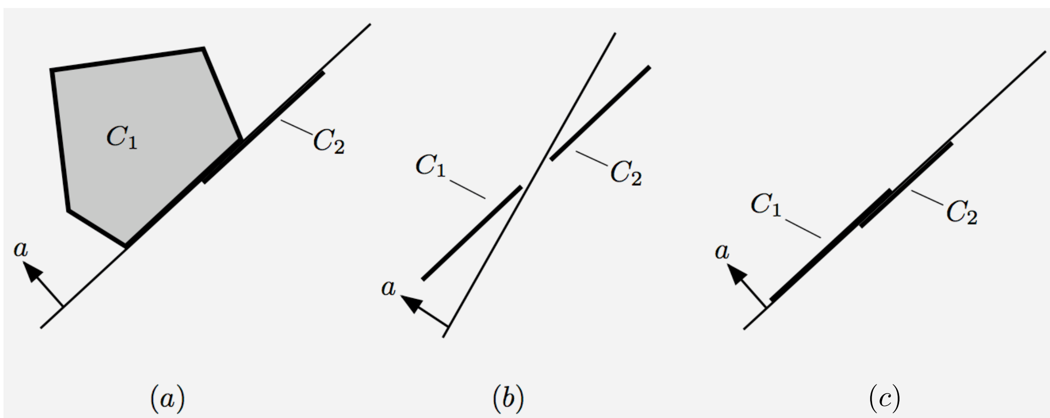
- Review of hyperplane separation
- Nonvertical hyperplanes
- Convex conjugate functions
- Conjugacy theorem
- Examples

Reading: Section 1.5, 1.6

ADDITIONAL THEOREMS

- **Fundamental Characterization**: The closure of the convex hull of a set $C \subset \Re^n$ is the intersection of the closed halfspaces that contain C. (Proof uses the strict separation theorem.)

- We say that a hyperplane *properly separates* C_1 *and* C_2 if it separates C_1 and C_2 and does not fully contain both C_1 and C_2.



- **Proper Separation Theorem**: Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. There exists a hyperplane that properly separates C_1 and C_2 if and only if

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$$

PROPER POLYHEDRAL SEPARATION

- Recall that two convex sets C and P such that

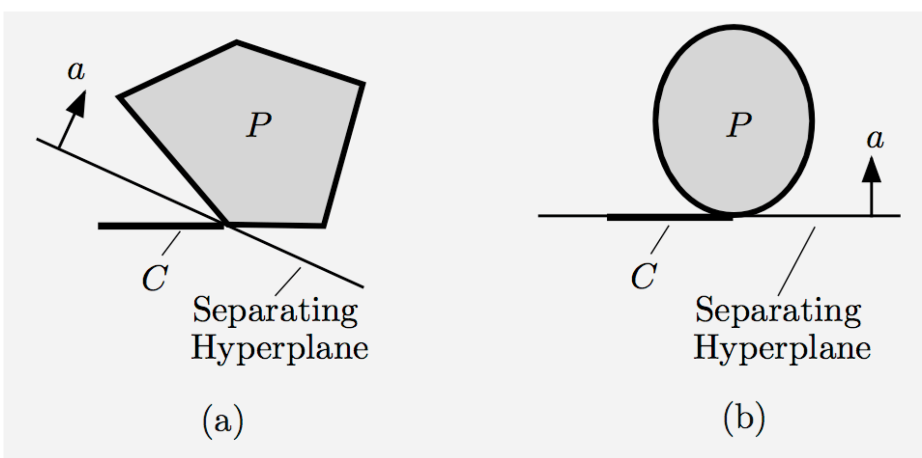
$$\mathrm{ri}(C) \cap \mathrm{ri}(P) = \emptyset$$

can be properly separated, i.e., by a hyperplane that does not contain both C and P.

- If P is polyhedral and the slightly stronger condition

$$\mathrm{ri}(C) \cap P = \emptyset$$

holds, then the properly separating hyperplane can be chosen so that it does not contain the nonpolyhedral set C while it may contain P.

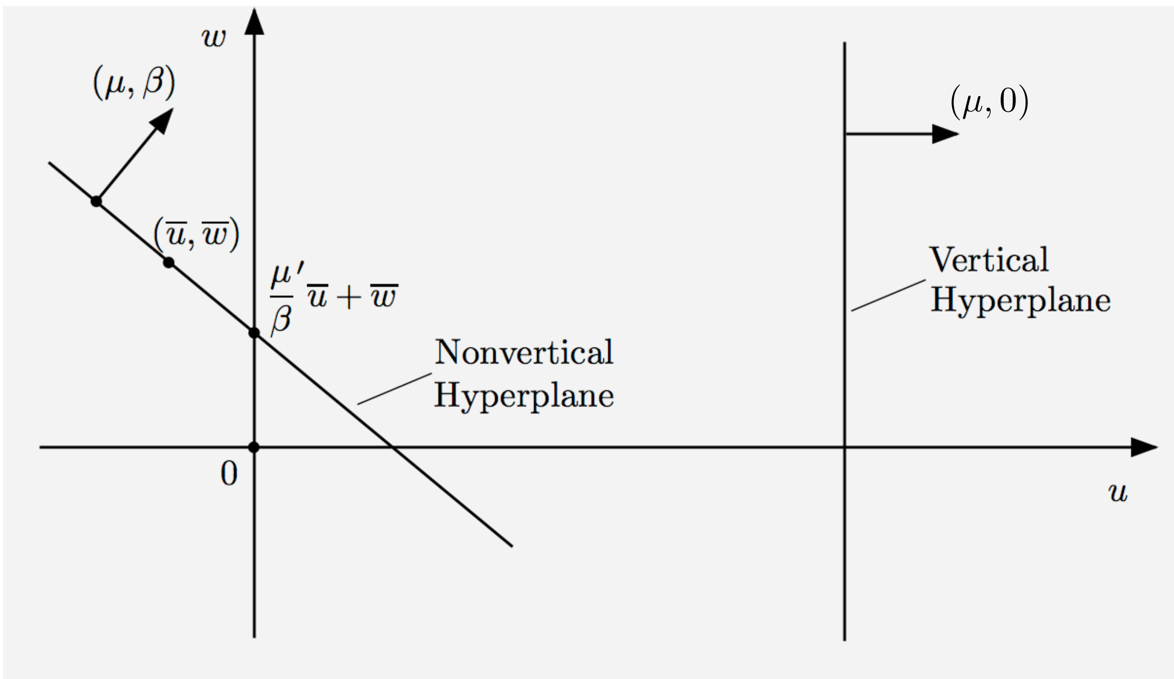


On the left, the separating hyperplane can be chosen so that it does not contain C. On the right where P is not polyhedral, this is not possible.

NONVERTICAL HYPERPLANES

- A hyperplane in $\Re^{n+1}$ with normal (μ, β) is nonvertical if $\beta \neq 0$.

- It intersects the $(n+1)$st axis at $\xi = (\mu/\beta)'\overline{u} + \overline{w}$, where $(\overline{u}, \overline{w})$ is any vector on the hyperplane.



- A nonvertical hyperplane that contains the epigraph of a function in its "upper" halfspace, provides lower bounds to the function values.

- The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the "upper" halfspace of some nonvertical hyperplane.

NONVERTICAL HYPERPLANE THEOREM

- Let C be a nonempty convex subset of $\Re^{n+1}$ that contains no vertical lines. Then:

  (a) C is contained in a closed halfspace of a non-vertical hyperplane, i.e., there exist $\mu \in \Re^n$, $\beta \in \Re$ with $\beta \neq 0$, and $\gamma \in \Re$ such that $\mu' u + \beta w \geq \gamma$ for all $(u, w) \in C$.

  (b) If $(\overline{u}, \overline{w}) \notin \mathrm{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(\overline{u}, \overline{w})$ and C.

Proof: Note that cl(C) contains no vert. line [since C contains no vert. line, ri(C) contains no vert. line, and ri(C) and cl(C) have the same recession cone]. So we just consider the case: C closed.

(a) C is the intersection of the closed halfspaces containing C. If all these corresponded to vertical hyperplanes, C would contain a vertical line.

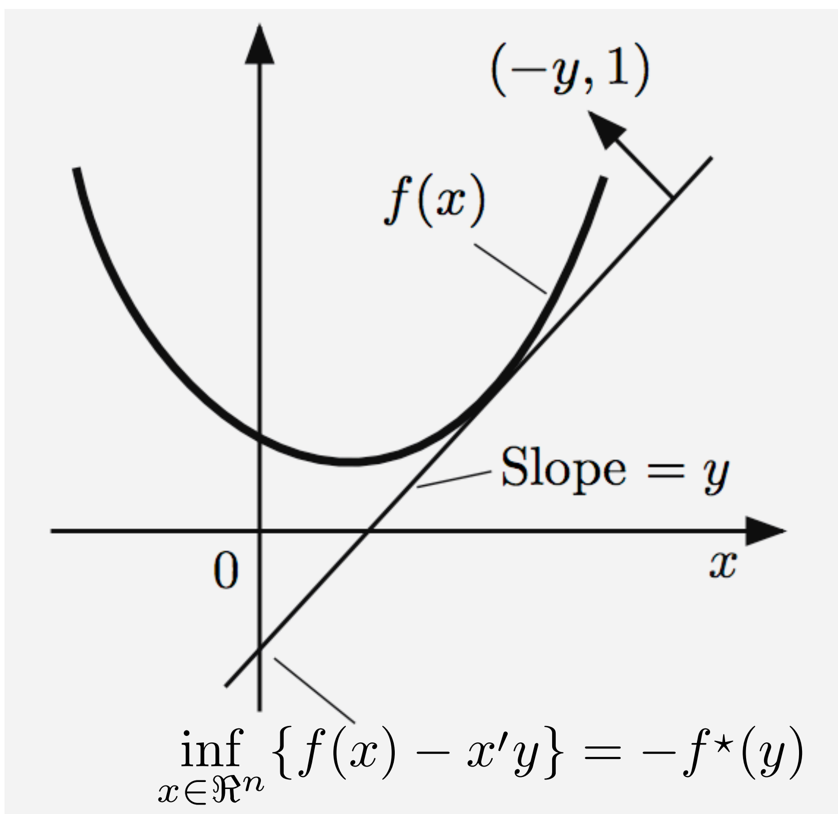
(b) There is a hyperplane strictly separating $(\overline{u}, \overline{w})$ and C. If it is nonvertical, we are done, so assume it is vertical. "Add" to this vertical hyperplane a small ϵ-multiple of a nonvertical hyperplane containing C in one of its halfspaces as per (a).

CONJUGATE CONVEX FUNCTIONS

- Consider a function f and its epigraph

Nonvertical hyperplanes supporting epi(f)
$\mapsto$ Crossing points of vertical axis

$$f^\star(y) = \sup_{x \in \Re^n} \big\{ x'y - f(x) \big\}, \qquad y \in \Re^n.$$

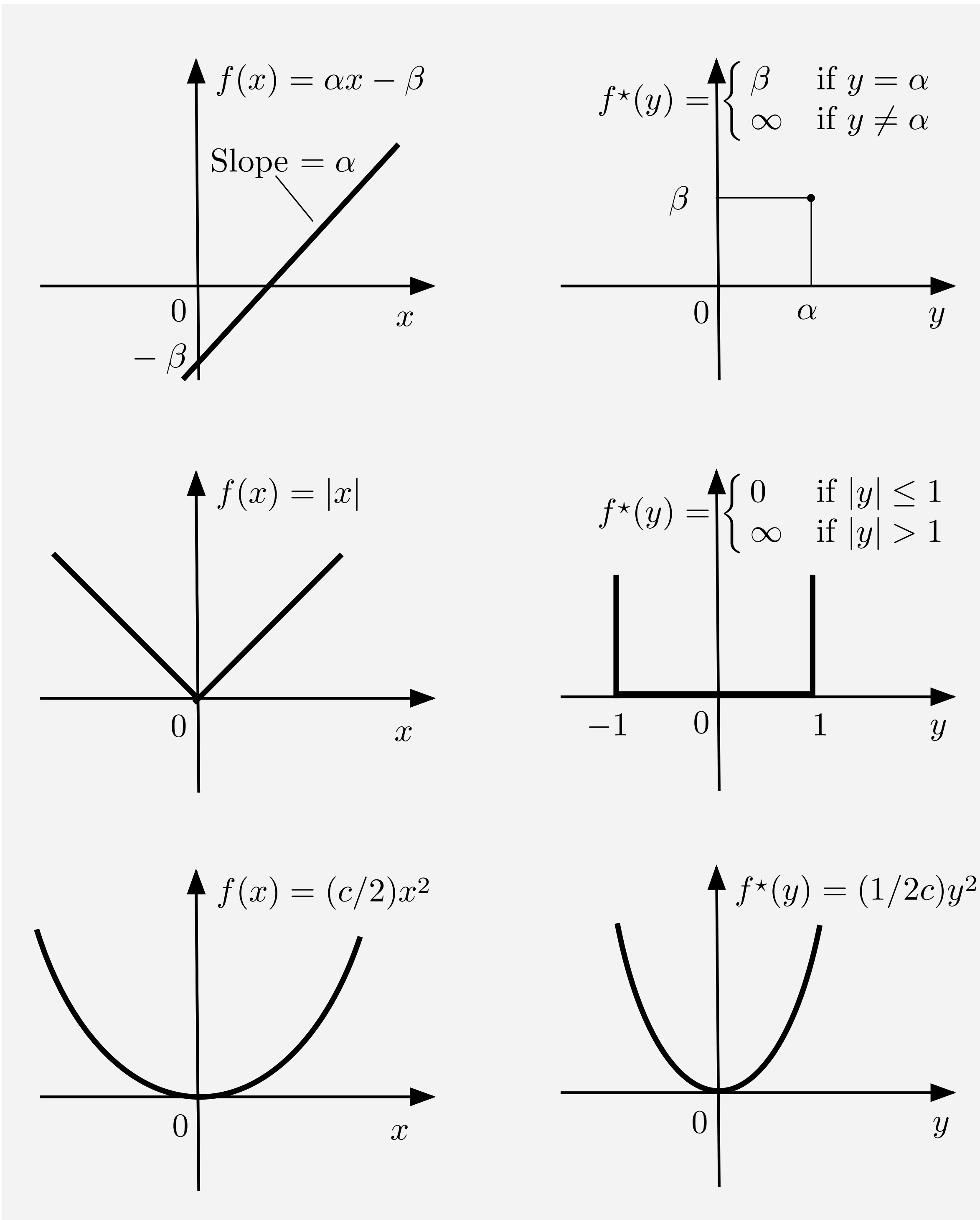


- For any $f : \Re^n \mapsto [-\infty, \infty]$, its *conjugate convex function* is defined by

$$f^\star(y) = \sup_{x \in \Re^n} \big\{ x'y - f(x) \big\}, \qquad y \in \Re^n$$

EXAMPLES

$$f^\star(y) = \sup_{x \in \Re^n} \{x'y - f(x)\}, \qquad y \in \Re^n$$



CONJUGATE OF CONJUGATE

- From the definition

$$f^\star(y) = \sup_{x\in\Re^n} \big\{x'y - f(x)\big\}, \qquad y \in \Re^n,$$

note that $f^\star$ *is convex and closed*.

- Reason: $\text{epi}(f^\star)$ is the intersection of the epigraphs of the linear functions of y

$$x'y - f(x)$$

as x ranges over $\Re^n$.

- Consider the conjugate of the conjugate:

$$f^{\star\star}(x) = \sup_{y\in\Re^n} \big\{y'x - f^\star(y)\big\}, \qquad x \in \Re^n.$$

- $f^{\star\star}$ is convex and closed.

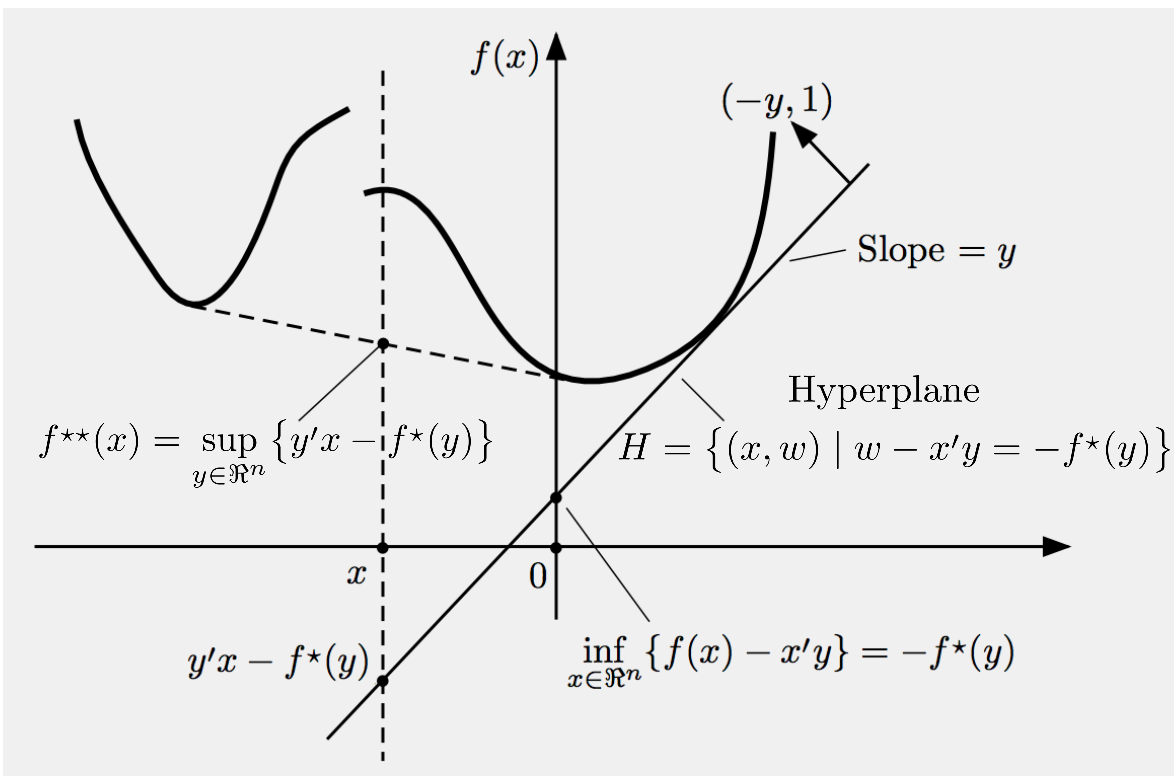
- **Important fact/Conjugacy theorem:** If f is closed proper convex, then $f^{\star\star} = f$.

CONJUGACY THEOREM - VISUALIZATION

$$f^\star(y) = \sup_{x \in \Re^n} \big\{ x'y - f(x) \big\}, \qquad y \in \Re^n$$

$$f^{\star\star}(x) = \sup_{y \in \Re^n} \big\{ y'x - f^\star(y) \big\}, \qquad x \in \Re^n$$

- If f is closed convex proper, then $f^{\star\star} = f$.



CONJUGACY THEOREM

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be a function, let $\check{\text{cl}}\, f$ be its convex closure, let $f^\star$ be its convex conjugate, and consider the conjugate of $f^\star$,

$$f^{\star\star}(x) = \sup_{y \in \Re^n} \big\{ y'x - f^\star(y) \big\}, \qquad x \in \Re^n$$

(a) We have

$$f(x) \geq f^{\star\star}(x), \qquad \forall\, x \in \Re^n$$

(b) If f is convex, then properness of any one of f, $f^\star$, and $f^{\star\star}$ implies properness of the other two.

(c) If f is closed proper and convex, then

$$f(x) = f^{\star\star}(x), \qquad \forall\, x \in \Re^n$$

(d) If $\check{\text{cl}}\, f(x) > -\infty$ for all $x \in \Re^n$, then

$$\check{\text{cl}}\, f(x) = f^{\star\star}(x), \qquad \forall\, x \in \Re^n$$

PROOF OF CONJUGACY THEOREM (A), (C)

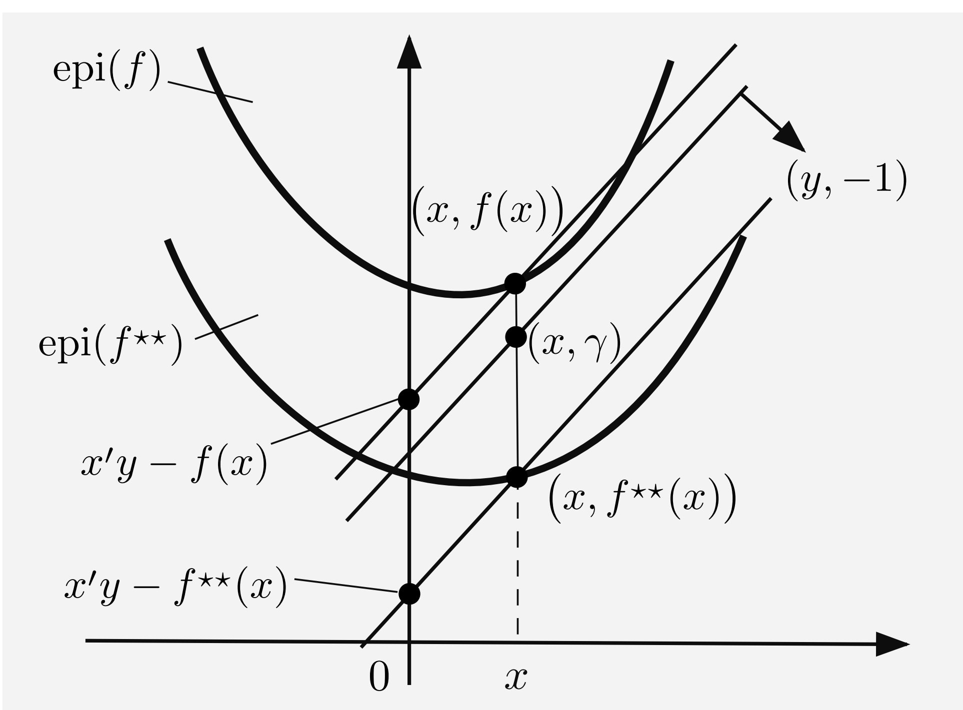
- (a) For all x, y, we have $f^\star(y) \geq y'x - f(x)$, implying that $f(x) \geq \sup_y\{y'x - f^\star(y)\} = f^{\star\star}(x)$.

- (c) By contradiction. Assume there is $(x, \gamma) \in \text{epi}(f^{\star\star})$ with $(x, \gamma) \notin \text{epi}(f)$. There exists a nonvertical hyperplane with normal $(y, -1)$ that strictly separates (x, γ) and $\text{epi}(f)$. (The vertical component of the normal vector is normalized to -1.)

- Consider two parallel hyperplanes, translated to pass through $\big(x, f(x)\big)$ and $\big(x, f^{\star\star}(x)\big)$. Their vertical crossing points are $x'y - f(x)$ and $x'y - f^{\star\star}(x)$, and lie strictly above and below the crossing point of the strictly sep. hyperplane. Hence

$$x'y - f(x) > x'y - f^{\star\star}(x)$$

which contradicts part (a). **Q.E.D.**



A COUNTEREXAMPLE

- A counterexample (with closed convex but improper f) showing the need to assume properness in order for $f = f^{\star\star}$:

$$f(x) = \begin{cases} \infty & \text{if } x > 0, \\ -\infty & \text{if } x \leq 0. \end{cases}$$

We have

$$f^{\star}(y) = \infty, \qquad \forall\, y \in \Re^n,$$

$$f^{\star\star}(x) = -\infty, \qquad \forall\, x \in \Re^n.$$

But

$$\check{\text{cl}}\, f = f,$$

so $\check{\text{cl}}\, f \neq f^{\star\star}$.