LECTURE 8

LECTURE OUTLINE

- Review of conjugate convex functions
- Min common/max crossing duality
- Weak duality
- Special cases

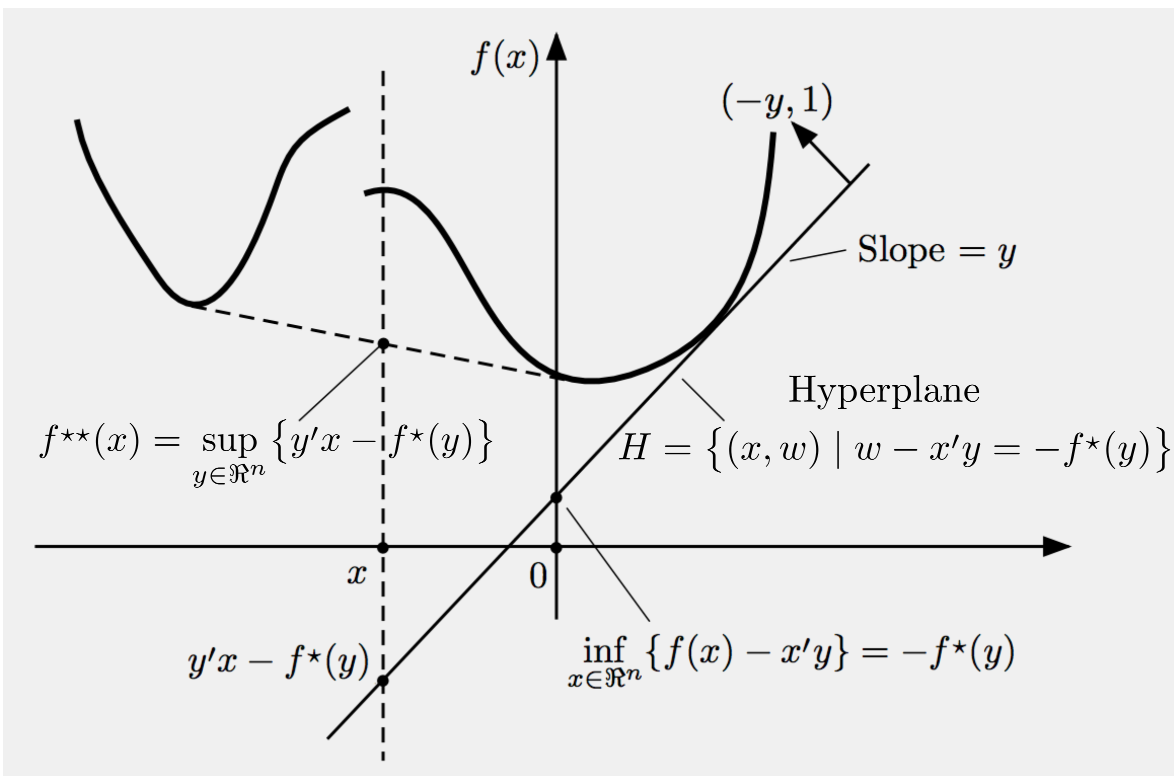
Reading: Sections 1.6, 4.1, 4.2

CONJUGACY THEOREM

$$f^\star(y) = \sup_{x \in \Re^n} \big\{ x'y - f(x) \big\}, \qquad y \in \Re^n$$

$$f^{\star\star}(x) = \sup_{y \in \Re^n} \big\{ y'x - f^\star(y) \big\}, \qquad x \in \Re^n$$

- If f is closed convex proper, then $f^{\star\star} = f$.



A FEW EXAMPLES

- l_p and l_q norm conjugacy, where $\frac{1}{p} + \frac{1}{q} = 1$

$$f(x) = \frac{1}{p} \sum_{i=1}^{n} |x_i|^p, \qquad f^\star(y) = \frac{1}{q} \sum_{i=1}^{n} |y_i|^q$$

- Conjugate of a strictly convex quadratic

$$f(x) = \frac{1}{2} x' Q x + a' x + b,$$

$$f^\star(y) = \frac{1}{2} (y - a)' Q^{-1} (y - a) - b.$$

- Conjugate of a function obtained by invertible linear transformation/translation of a function p

$$f(x) = p\big(A(x - c)\big) + a' x + b,$$

$$f^\star(y) = q\big((A')^{-1}(y - a)\big) + c' y + d,$$

where q is the conjugate of p and $d = -(c' a + b)$.

SUPPORT FUNCTIONS

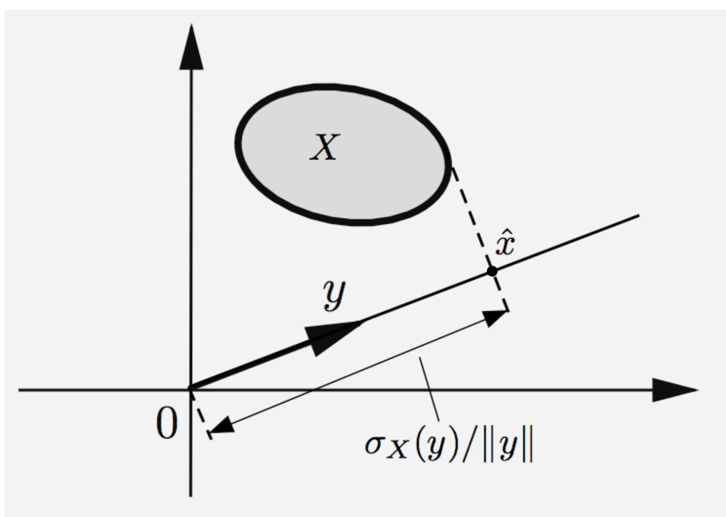
- Conjugate of indicator function δ_X of set X

$$\sigma_X(y) = \sup_{x \in X} y'x$$

is called the *support function of* X.

- To determine $\sigma_X(y)$ for a given vector y, we project the set X on the line determined by y, we find $\hat{x}$, the extreme point of projection in the direction y, and we scale by setting

$$\sigma_X(y) = \|\hat{x}\| \cdot \|y\|$$



- $\mathrm{epi}(\sigma_X)$ is a closed convex cone.

- The sets X, $\mathrm{cl}(X)$, $\mathrm{conv}(X)$, and $\mathrm{cl}\big(\mathrm{conv}(X)\big)$ all have the same support function (by the conjugacy theorem).

SUPPORT FN OF A CONE - POLAR CONE

- The conjugate of the indicator function δ_C is the support function, $\sigma_C(y) = \sup_{x \in C} y'x$.
- If C is a cone,

$$\sigma_C(y) = \begin{cases} 0 & \text{if } y'x \leq 0, \ \forall \ x \in C, \\ \infty & \text{otherwise} \end{cases}$$

i.e., σ_C is the indicator function δ_{C^*} of the cone

$$C^* = \{y \mid y'x \leq 0, \ \forall \ x \in C\}$$

This is called the *polar cone of* C.

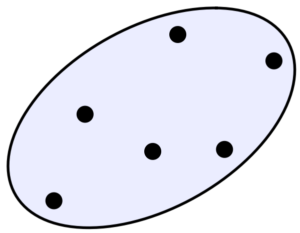
- By the Conjugacy Theorem the polar cone of C^* is $\text{cl}\big(\text{conv}(C)\big)$. This is the *Polar Cone Theorem.*
- **Special case:** If $C = \text{cone}\big(\{a_1, \ldots, a_r\}\big)$, then

$$C^* = \{x \mid a_j'x \leq 0, \ j = 1, \ldots, r\}$$

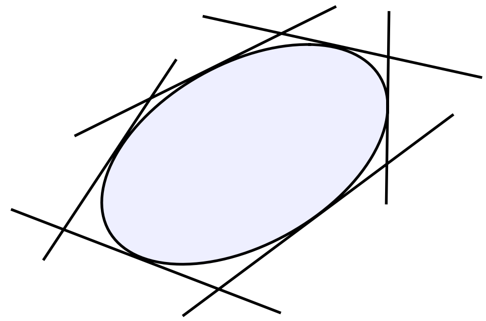
- **Farkas' Lemma:** $(C^*)^* = C$.
- True because C is a closed set $\big[\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ is the image of the positive orthant $\{\alpha \mid \alpha \geq 0\}$ under the linear transformation that maps α to $\sum_{j=1}^r \alpha_j a_j\big]$, and the image of any polyhedral set under a linear transformation is a closed set.

EXTENDING DUALITY CONCEPTS

- From **dual descriptions of sets**

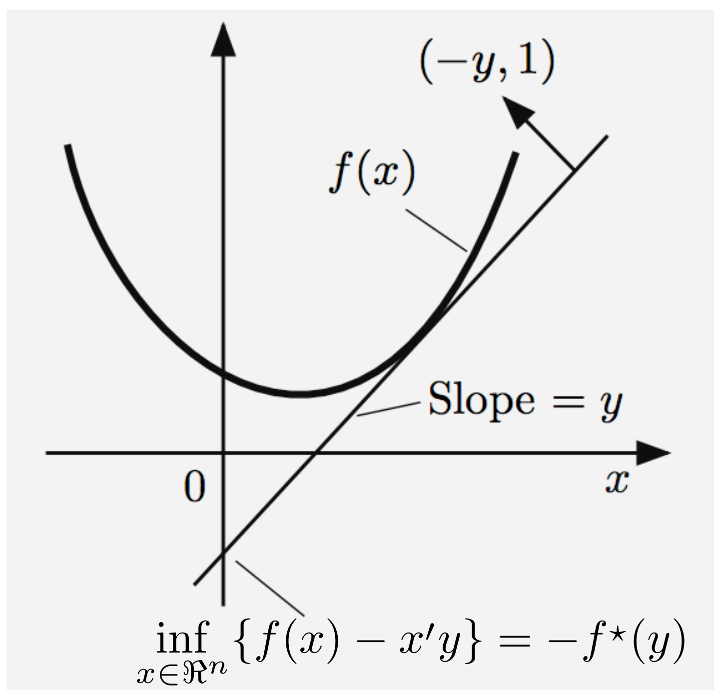


A union of points



An intersection of halfspaces

- To **dual descriptions of functions** (applying set duality to epigraphs)



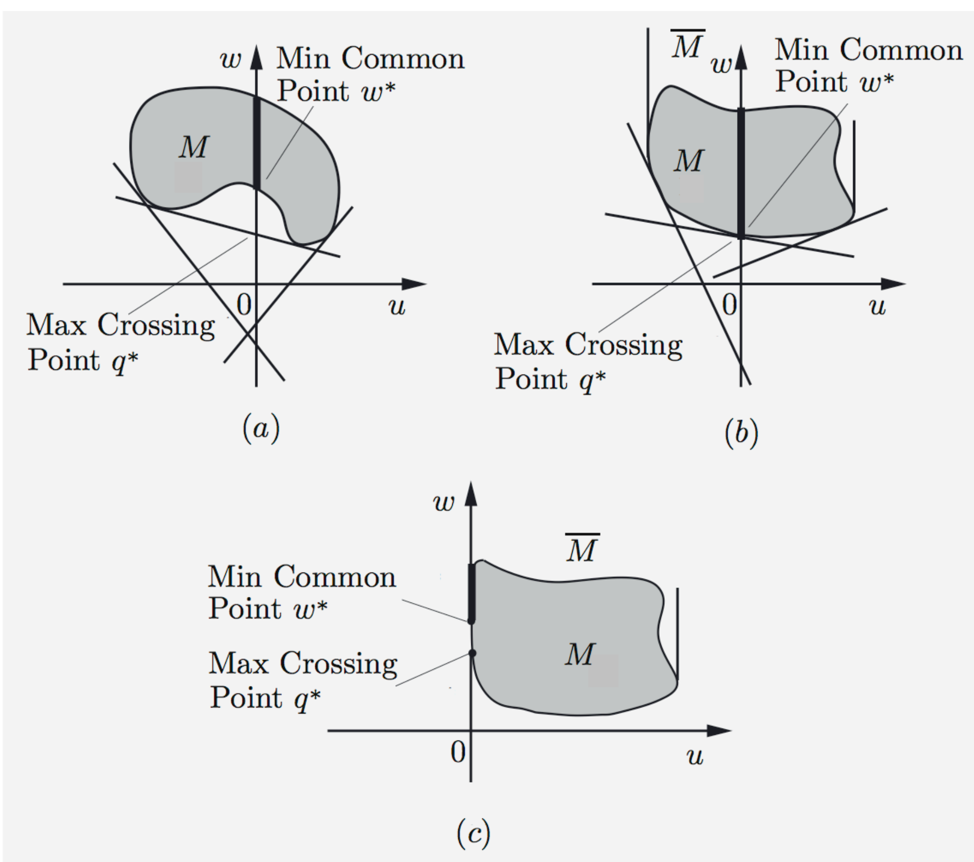
- We now go to **dual descriptions of problems**, by applying conjugacy constructions to a simple generic geometric optimization problem

MIN COMMON / MAX CROSSING PROBLEMS

- We introduce a pair of fundamental problems:
- Let M be a nonempty subset of $\Re^{n+1}$

(a) *Min Common Point Problem*: Consider all vectors that are common to M and the $(n+1)$st axis. Find one whose $(n+1)$st component is minimum.

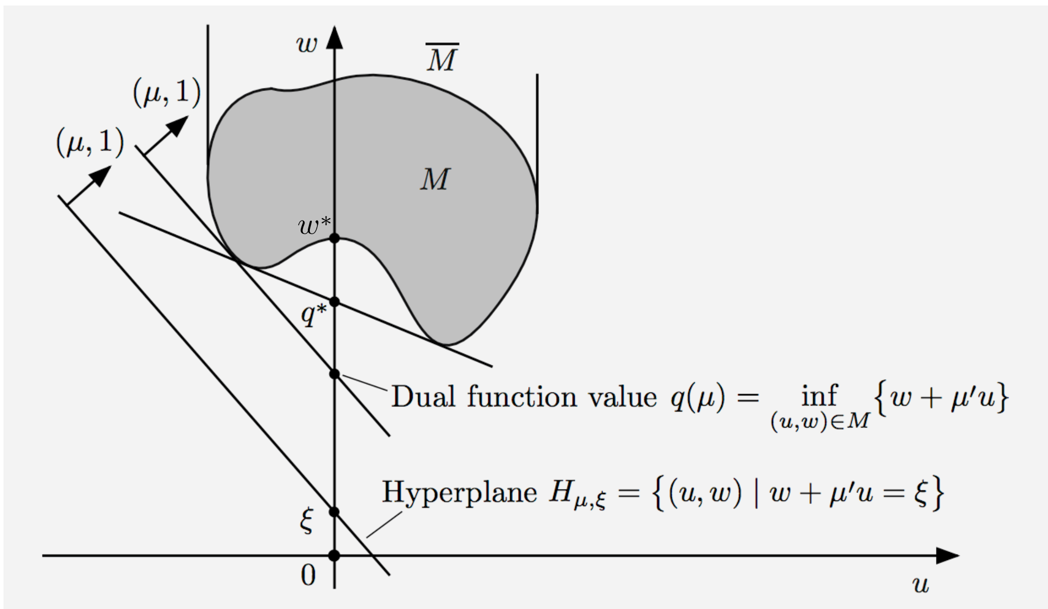
(b) *Max Crossing Point Problem*: Consider nonvertical hyperplanes that contain M in their "upper" closed halfspace. Find one whose crossing point of the $(n+1)$st axis is maximum.



MATHEMATICAL FORMULATIONS

- **Optimal value of min common problem:**

$$w^* = \inf_{(0,w)\in M} w$$



- **Math formulation of max crossing problem:** Focus on hyperplanes with normals $(\mu, 1)$ whose crossing point ξ satisfies

$$\xi \le w + \mu' u, \qquad \forall\ (u, w) \in M$$

Max crossing problem is to maximize ξ subject to $\xi \le \inf_{(u,w)\in M}\{w + \mu' u\}$, $\mu \in \Re^n$, or

$$\text{maximize} \;\; q(\mu) \stackrel{\triangle}{=} \inf_{(u,w)\in M}\{w + \mu' u\}$$

$$\text{subject to} \;\; \mu \in \Re^n.$$

GENERIC PROPERTIES – WEAK DUALITY

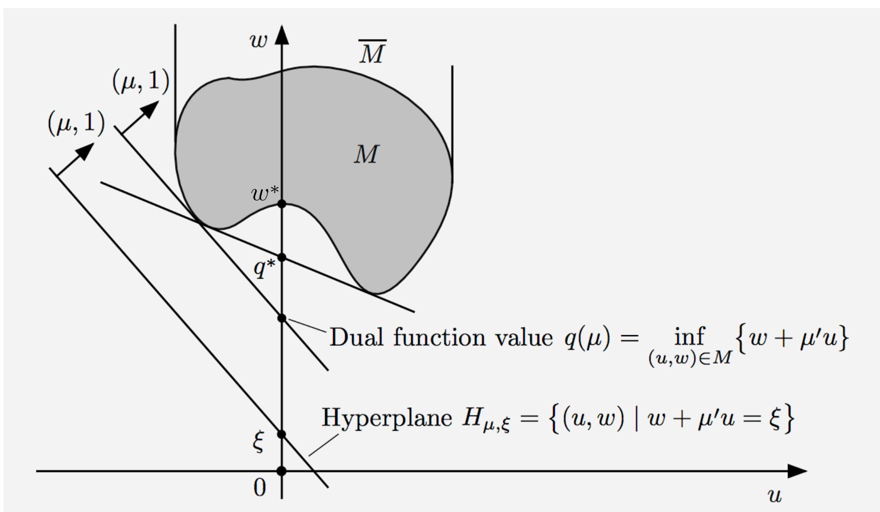
- Min common problem

$$\inf_{(0,w)\in M} w$$

- Max crossing problem

$$\text{maximize} \;\; q(\mu) \stackrel{\triangle}{=} \inf_{(u,w)\in M} \{w + \mu' u\}$$

$$\text{subject to} \;\; \mu \in \Re^n.$$



- Note that q is concave and upper-semicontinuous (inf of linear functions).

- **Weak Duality:** For all $\mu \in \Re^n$

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} \leq \inf_{(0,w)\in M} w = w^*,$$

so maximizing over $\mu \in \Re^n$, we obtain $q^* \leq w^*$.

- We say that **strong duality** holds if $q^* = w^*$.

CONNECTION TO CONJUGACY

- **An important special case:**

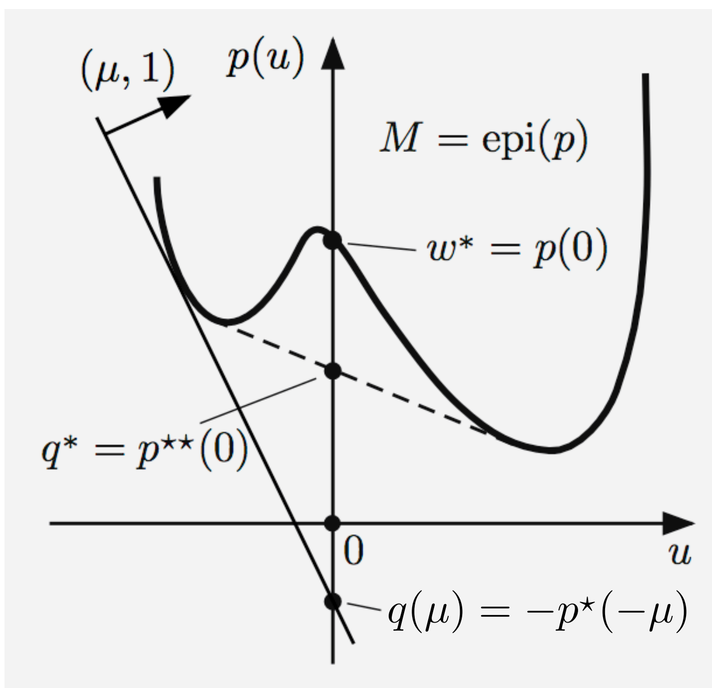
$$M = \mathrm{epi}(p)$$

where $p : \Re^n \mapsto [-\infty, \infty]$. Then $w^* = p(0)$, and

$$q(\mu) = \inf_{(u,w)\in \mathrm{epi}(p)} \{w+\mu' u\} = \inf_{\{(u,w)\mid p(u)\le w\}} \{w+\mu' u\},$$

and finally

$$q(\mu) = \inf_{u\in\Re^m} \big\{p(u) + \mu' u\big\}$$



- Thus, $q(\mu) = -p^\star(-\mu)$ and

$$q^* = \sup_{\mu\in\Re^n} q(\mu) = \sup_{\mu\in\Re^n} \big\{0\cdot(-\mu) - p^\star(-\mu)\big\} = p^{\star\star}(0)$$

GENERAL OPTIMIZATION DUALITY

- Consider minimizing a function $f : \Re^n \mapsto [-\infty, \infty]$.
- Let $F : \Re^{n+r} \mapsto [-\infty, \infty]$ be a function with

$$f(x) = F(x, 0), \qquad \forall\, x \in \Re^n$$

- Consider the *perturbation function*

$$p(u) = \inf_{x \in \Re^n} F(x, u)$$

and the MC/MC framework with $M = \text{epi}(p)$

- The min common value w^* is

$$w^* = p(0) = \inf_{x \in \Re^n} F(x, 0) = \inf_{x \in \Re^n} f(x)$$

- The dual function is

$$q(\mu) = \inf_{u \in \Re^r} \big\{p(u) + \mu' u\big\} = \inf_{(x,u) \in \Re^{n+r}} \big\{F(x, u) + \mu' u\big\}$$

so $q(\mu) = -F^\star(0, -\mu)$, where $F^\star$ is the conjugate of F, viewed as a function of (x, u)

- We have

$$q^* = \sup_{\mu \in \Re^r} q(\mu) = -\inf_{\mu \in \Re^r} F^\star(0, -\mu) = -\inf_{\mu \in \Re^r} F^\star(0, \mu),$$

and weak duality has the form

$$w^* = \inf_{x \in \Re^n} F(x, 0) \geq -\inf_{\mu \in \Re^r} F^\star(0, \mu) = q^*$$

CONSTRAINED OPTIMIZATION

- Minimize $f : \Re^n \mapsto \Re$ over the set

$$C = \{x \in X \mid g(x) \leq 0\},$$

where $X \subset \Re^n$ and $g : \Re^n \mapsto \Re^r$.

- Introduce a "perturbed constraint set"

$$C_u = \{x \in X \mid g(x) \leq u\}, \qquad u \in \Re^r,$$

and the function

$$F(x,u) = \begin{cases} f(x) & \text{if } x \in C_u, \\ \infty & \text{otherwise,} \end{cases}$$

which satisfies $F(x,0) = f(x)$ for all $x \in C$.

- Consider *perturbation function*

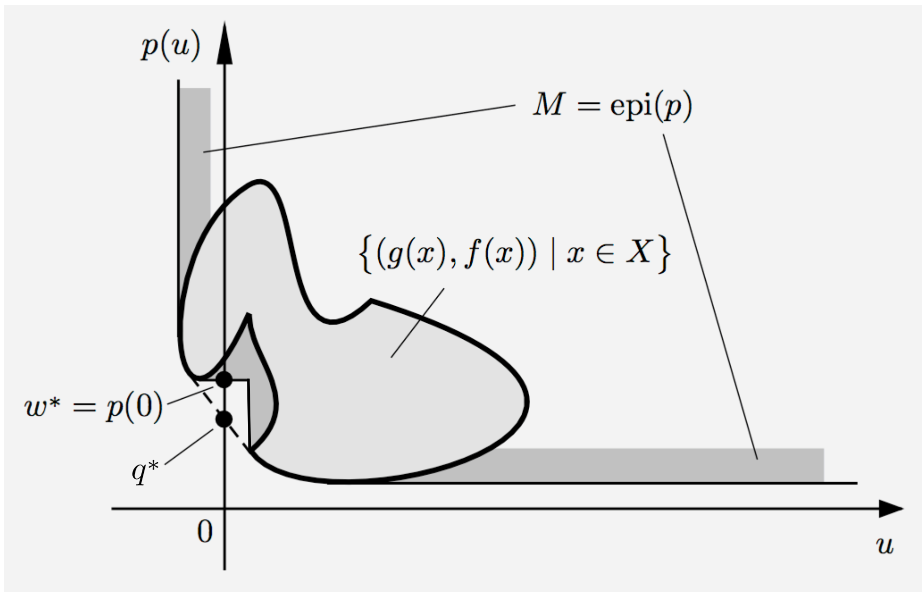
$$p(u) = \inf_{x \in \Re^n} F(x,u) = \inf_{x \in X,\, g(x) \leq u} f(x),$$

and the MC/MC framework with $M = \text{epi}(p)$.

CONSTR. OPT. - PRIMAL AND DUAL FNS

- Perturbation function (or *primal function*)

$$p(u) = \inf_{x \in X,\, g(x) \le u} f(x),$$



- Introduce $L(x, \mu) = f(x) + \mu' g(x)$. Then

$$\begin{aligned} q(\mu) &= \inf_{u \in \Re^r} \big\{ p(u) + \mu' u \big\} \\ &= \inf_{u \in \Re^r,\, x \in X,\, g(x) \le u} \big\{ f(x) + \mu' u \big\} \\ &= \begin{cases} \inf_{x \in X} L(x, \mu) & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

LINEAR PROGRAMMING DUALITY

- Consider the linear program

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad a_j'x \geq b_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $c \in \Re^n$, $a_j \in \Re^n$, and $b_j \in \Re$, $j = 1, \ldots, r$.

- For $\mu \geq 0$, the dual function has the form

$$\begin{aligned} q(\mu) &= \inf_{x \in \Re^n} L(x, \mu) \\ &= \inf_{x \in \Re^n} \left\{ c'x + \sum_{j=1}^{r} \mu_j (b_j - a_j'x) \right\} \\ &= \begin{cases} b'\mu & \text{if } \sum_{j=1}^{r} a_j \mu_j = c, \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

- Thus the dual problem is

$$\begin{aligned} &\text{maximize} \quad b'\mu \\ &\text{subject to} \quad \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0. \end{aligned}$$