Mortgage Interest & Principal Payments

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A few years ago, I bought a house, and got frustrated when the lending representatives were unable to explain how they had arrived at particular mortgage payments (all they could do was table lookups). I was curious about how the amount of the fixed monthly payment is determined, and how it gets partitioned into principal and interest – so I derived it directly, as follows.

Suppose we wish to take a loan of $A$ dollars at yearly interest rate $R$ (equivalently, monthly interest rate $M = R/12$) to be paid monthly over $Y$ years (in $N$ monthly payments, where $N = Y \cdot 12$). The object of this problem is to determine the (constant) total monthly payment $T$, and its (varying) apportionment between principal $P_k$ and interest $I_k$ (for the $k$-th monthly payment).

**Problem:**
Express $T$, $P_k$, and $I_k$ as functions of $M$, $N$, and the loan amount $A$.

**Solution:**
Define the balance after the $k^{th}$ payment as $B_k$. Then

$$B_0 = A \quad \text{and} \quad B_N = 0$$

Moreover, the first interest payment will be on the entire loan amount:

$$I_0 = MA$$

and the sum of all the principal payments must equal the loan amount:

$$\sum_{k=1}^{N} P_k = A$$

The monthly payment $T$ is constant; thus $T_{k-1} = T_k$ for any $k > 1$, and

$$M(A - \sum_{j=1}^{k-2} P_j) + P_{k-1} = M(A - \sum_{j=1}^{k-1} P_j) + P_k$$

Cancelling common terms yields

$$P_k = (1 + M)P_{k-1}$$

so that

$$P_N = (1 + M)^{N-1}P_1$$
Now, consider the initial balance \( A \) and sequence of \( N \) payments

\[
\begin{array}{|c|c|c|c|c|}
\hline
k & I_k & P_k & T & B_k \\
\hline
0 & & & A & \\
1 & MA & P_1 & MA + P_1 & A - P_1 \\
2 & M(A - P_1) & P_2 & MA - MP_1 + P_2 & A - P_1 - P_2 \\
\vdots & & & & \\
N & M(A - \sum_{k=1}^{N-1} P_k) & P_N & M(A - \sum_{k=1}^{N-1} P_k) + P_N & A - \sum_{k=1}^{N} P_k \\
\hline
\end{array}
\]

Since \( \sum_{k=1}^{N} P_k = A \), we can rewrite the last row (i.e., the \( N^{th} \) payment) as

\[
\begin{array}{|c|c|c|c|c|}
\hline
k & I_N & P_N & T & B_N \\
\hline
N & MP_N & P_N & (1 + M)P_N & 0 \\
\hline
\end{array}
\]

Equating \( T = T_1 = T_N \) and substituting for \( P_N \) yields

\[
P_1 = \frac{MA}{(1 + M)^N - 1}
\]

We can now – as specified by the problem statement – express \( T, P_k, \) and \( I_k \) solely as functions of \( M, N, \) and \( A \):

\[
T = (1 + M)^N P_1 = \frac{MA(1 + M)^N}{(1 + M)^N - 1}
\]

\[
P_k = (1 + M)^{k-1} P_1 = (1 + M)^{k-1} \frac{MA}{(1 + M)^N - 1}
\]

\[
I_k = T - P_k
\]

Lenders, to avoid evaluating so many exponentiations, probably compute mortgage tables using the above closed-form expression for \( T \), the recurrence

\[
B_0 = A; \quad I_k = MB_{k-1}; \quad P_k = T - I_k; \quad B_k = B_{k-1} - P_k
\]

and some rule for rounding fractional values to whole pennies.

(Note: see C code mortgage.c for an implementation.)