Mortgage Interest & Principal Payments

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A few years ago, I bought a house, and got frustrated when the lending representatives were unable to explain how they had arrived at particular mortgage payments (all they could do was table lookups). I was curious about how the amount of the fixed monthly payment is determined, and how it gets partitioned into principal and interest – so I derived it directly, as follows.

Suppose we wish to take a loan of A dollars at yearly interest rate R (equivalently, monthly interest rate $M = R/12$) to be paid monthly over Y years (in N monthly payments, where $N = Y \cdot 12$). The object of this problem is to determine the (constant) total monthly payment T, and its (varying) apportionment between principal P_k and interest I_k (for the k-th monthly payment).

Problem:

Express T, P_k, and I_k as functions of M, N, and the loan amount A.

Solution:

Define the balance after the k^{th} payment as B_k. Then

$$B_0 = A \qquad \text{and} \qquad B_N = 0$$

Moreover, the first interest payment will be on the entire loan amount:

$$I_0 = MA$$

and the sum of all the principal payments must equal the loan amount:

$$\sum_{k=1}^{N} P_k = A$$

The monthly payment T is constant; thus $T_{k-1} = T_k$ for any $k > 1$, and

$$M(A - \sum_{j=1}^{k-2} P_j) + P_{k-1} = M(A - \sum_{j=1}^{k-1} P_j) + P_k$$

Cancelling common terms yields

$$P_k = (1 + M)P_{k-1}$$

so that

$$P_N = (1 + M)^{N-1} P_1$$

Now, consider the initial balance A and sequence of N payments

k	I_k	P_k	T	B_k
0				A
1	MA	P_1	$MA + P_1$	$A - P_1$
2	$M(A - P_1)$	P_2	$MA - MP_1 + P_2$	$A - P_1 - P_2$
$\cdots$				
N	$M(A - \sum_{k=1}^{N-1} P_k)$	P_N	$M(A - \sum_{k=1}^{N-1} P_k) + P_N$	$A - \sum_{k=1}^{N} P_k$

Since $\sum_{k=1}^{N} P_k = A$, we can rewrite the last row (i.e., the N^{th} payment) as

k	I_N	P_N	T	B_N
N	MP_N	P_N	$(1 + M)P_N$	0

Equating $T = T_1 = T_N$ and substituting for P_N yields

$$P_1 = \frac{MA}{(1+M)^N - 1}$$

We can now – as specified by the problem statement – express T, P_k, and I_k solely as functions of M, N, and A:

$$T = (1+M)^N P_1 = \frac{MA(1+M)^N}{(1+M)^N - 1}$$

$$P_k = (1+M)^{k-1} P_1 = (1+M)^{k-1} \frac{MA}{(1+M)^N - 1}$$

$$I_k = T - P_k$$

Lenders, to avoid evaluating so many exponentiations, probably compute mortgage tables using the above closed-form expression for T, the recurrence

$$B_0 = A; \quad I_k = MB_{k-1}; \quad P_k = T - I_k; \quad B_k = B_{k-1} - P_k$$

and some rule for rounding fractional values to whole pennies.

(Note: see `C` code `mortgage.c` for an implementation.)