A Conservative $O(mnd)$-Time Infeasibility Test
for Linear Programs Posed Within a Convex Region

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The following is a simple observation leading to a “fast infeasibility test”
for linear programs posed inside convex regions for which a vertex description
is known. For a set of $m$ constraints and $n$ vertices in $d$ dimensions, the test
requires $O(m \cdot n \cdot d)$ time.

The test is conservative; if it succeeds, the posed LP is known to be infeasible
within the specified region. If the test fails, the LP must be solved using other
means.

**Theorem.** Given $n$ points $v_j$, $1 \leq j \leq n$, and $m$ hyperplanes $h_k$, $1 \leq k \leq m$, such that $\forall j, \sum_{k=1}^{m} (h_k \cdot v_j) < 0$, there exists no point $p$ in $\text{conv}(v_j)$ such that
$\forall k, h_k \cdot p \geq 0$. (Here, the inner product represents the signed distance between
a point and plane.)

![Diagram](image)

**Proof** (by contradiction). Suppose there exists a point $p$ in $\text{conv}(v_j)$ such that
$\forall k, h_k \cdot p \geq 0$.

By convexity,

$$p = \sum_{j=1}^{n} c_j v_j; \quad \forall j, c_j \geq 0; \quad \text{and} \quad \sum_{j=1}^{n} c_j = 1.$$ Substituting and summing over $k$,

$$\sum_{k=1}^{m} h_k \cdot \sum_{j=1}^{n} c_j v_j \geq 0.$$ Moving the inner product inside the $j$ summation,

$$\sum_{k=1}^{m} \sum_{j=1}^{n} c_j (h_k \cdot v_j) \geq 0,$$ and exchanging summation order yields

$$\sum_{j=1}^{n} c_j \left( \sum_{k=1}^{m} (h_k \cdot v_j) \right) \geq 0,$$ a contradiction, since the $c_j$ are nonnegative and not all zero. \qed

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