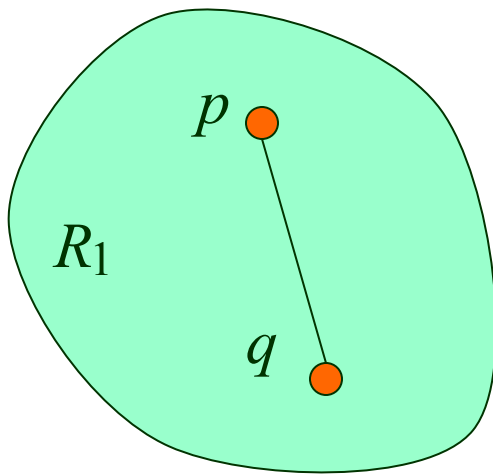


This is a PDF Version of Slides by:
Professor Soma Chaudhuri
Com S 311, Iowa State University
<http://www.cs.iastate.edu/~cs311/>

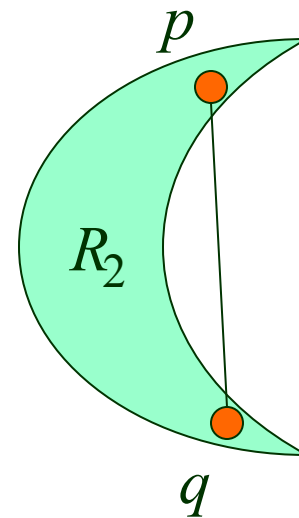
Convex Sets & Concave Sets

A planar region R is called **convex** if and only if for any pair of points p , q in R , the line segment \overline{pq} lies *completely* in R .

Otherwise, it is called **concave**.

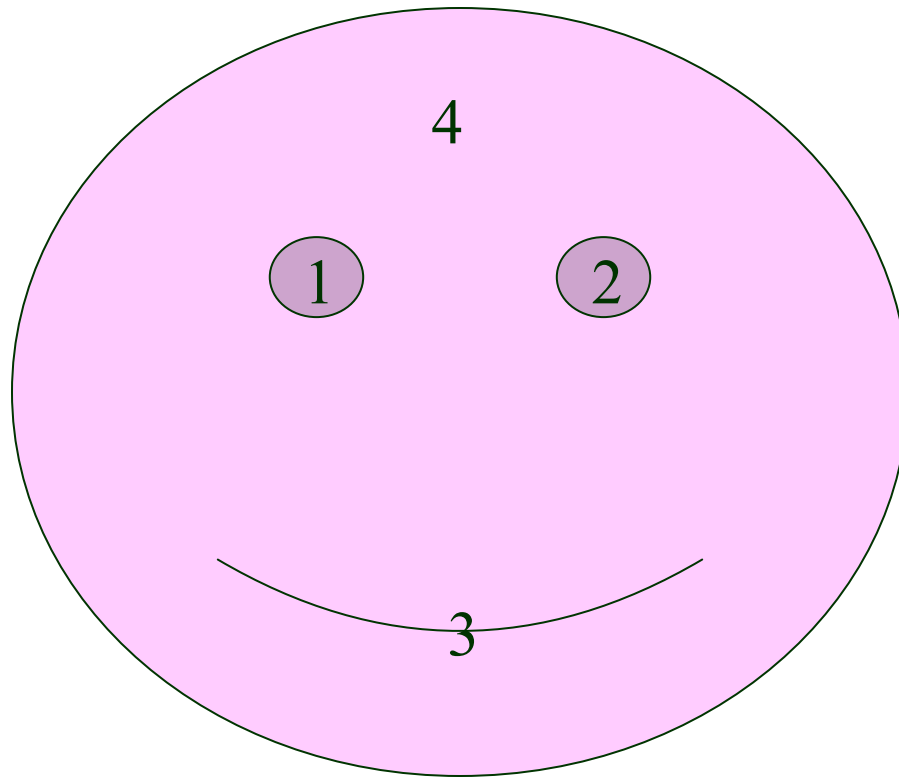


Convex



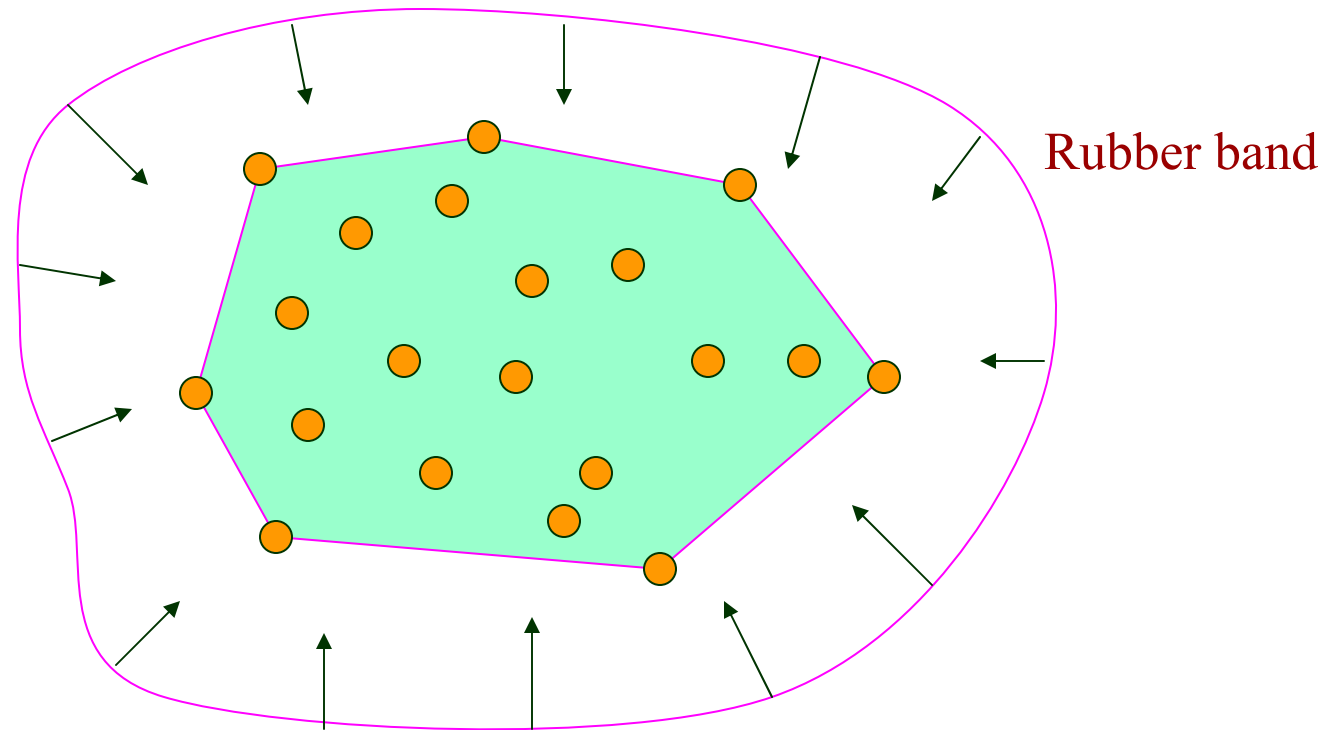
Concave

An Example



Convex Hull

The **convex hull** $CH(Q)$ of a set Q is the *smallest* convex region that contains Q .



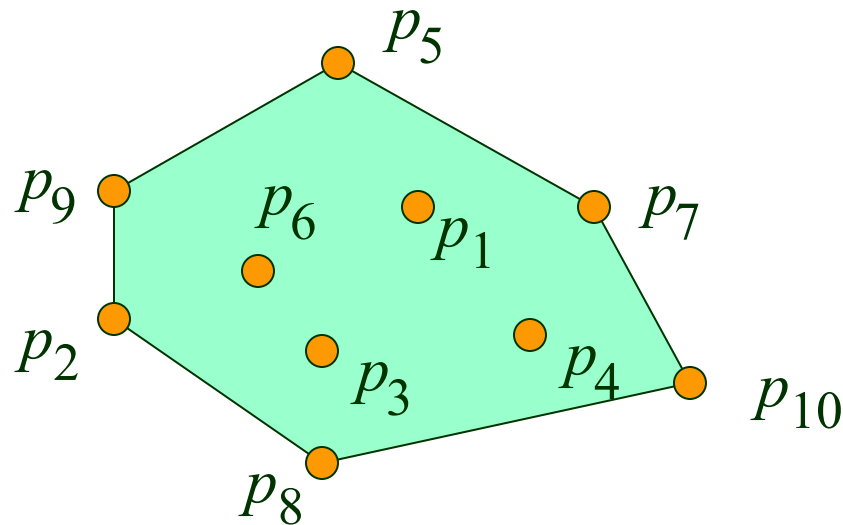
When Q is finite, its convex hull is the unique *convex polygon* whose vertices are from Q and that contains all points of Q .

The Convex Hull Problem

Input: a set $P = \{p_1, p_2, \dots, p_n\}$ of points

Output: a list of vertices of $\text{CH}(P)$ in counterclockwise order.

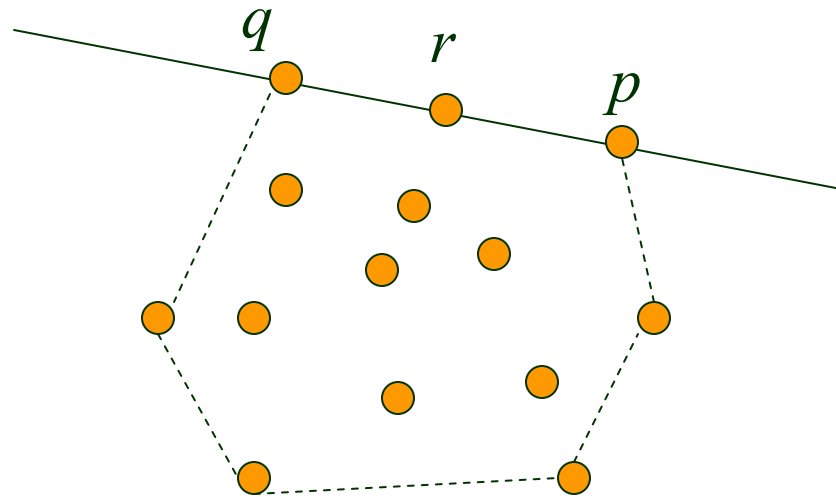
Example



Output: $p_5, p_9, p_2, p_8, p_{10}, p_7$.

Edges of a Convex Hull

- ✱ For every edge both endpoints $p, q \in P$.
- ✱ All other points in P lie
 - ✧ to the same side of the line passing through p and q , or
 - ✧ on the line segment \overline{pq} .



A Slow Convex Hull Algorithm

Slow-Convex-Hull(P)

$E \leftarrow \{\}$ // set of directed edges of CH(P) that bounds the
// points of P on the right.

for every ordered pair (p, q) , where $p, q \in P$ and $p \neq q$ // $\Theta(n^2)$ pairs

do valid \leftarrow true

for every point $r \neq p$ or q // $n - 2$ such points

do if r lies to the right of \vec{pq} or collinear with p and q
but not on \overline{pq}

then valid \leftarrow false

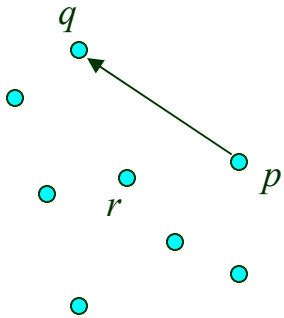
if valid

then $E \leftarrow E \cup \{\vec{pq}\}$ // \vec{pq} and \vec{qp} cannot be both in E

From E construct a list L of vertices of CH(P), sorted in

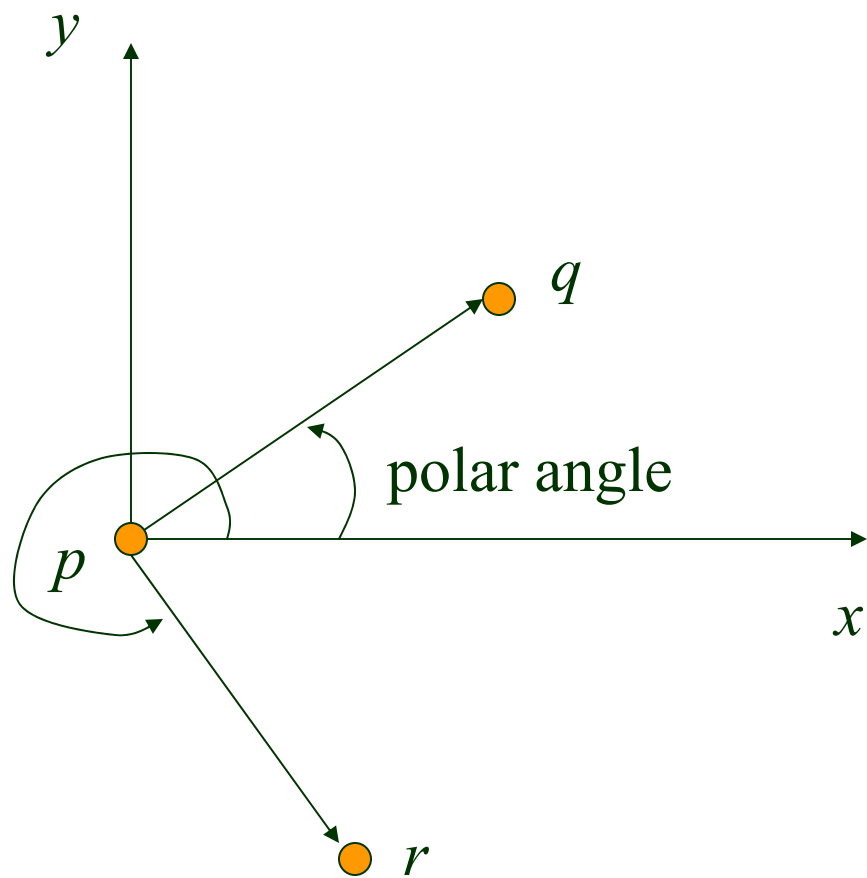
counterclockwise order. // $O(n^2)$ easily improvable to $O(n \lg n)$

return L



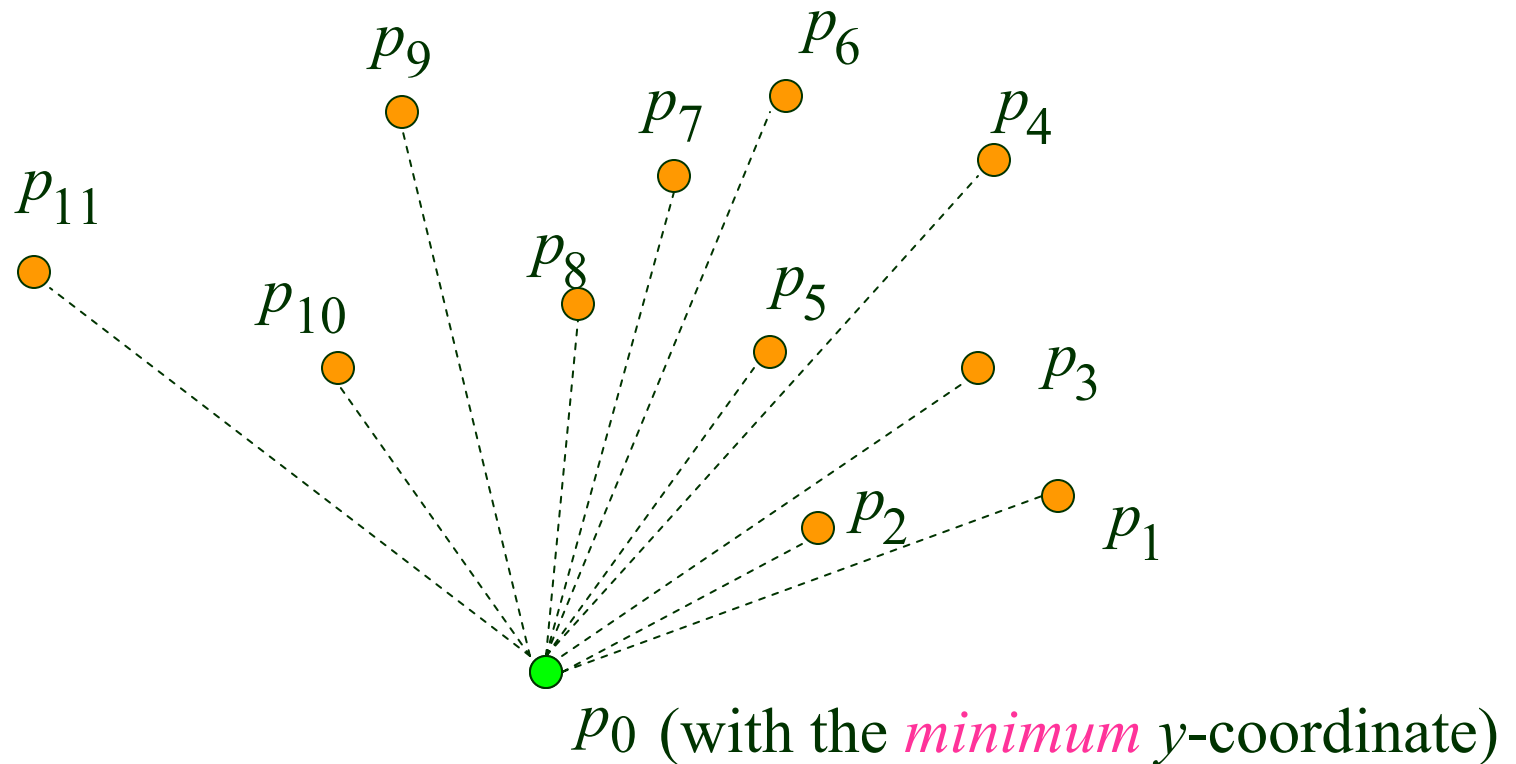
Running time $\Theta(n^3)$

Polar Angle



An Example of Graham Scan

Sort by polar angle.

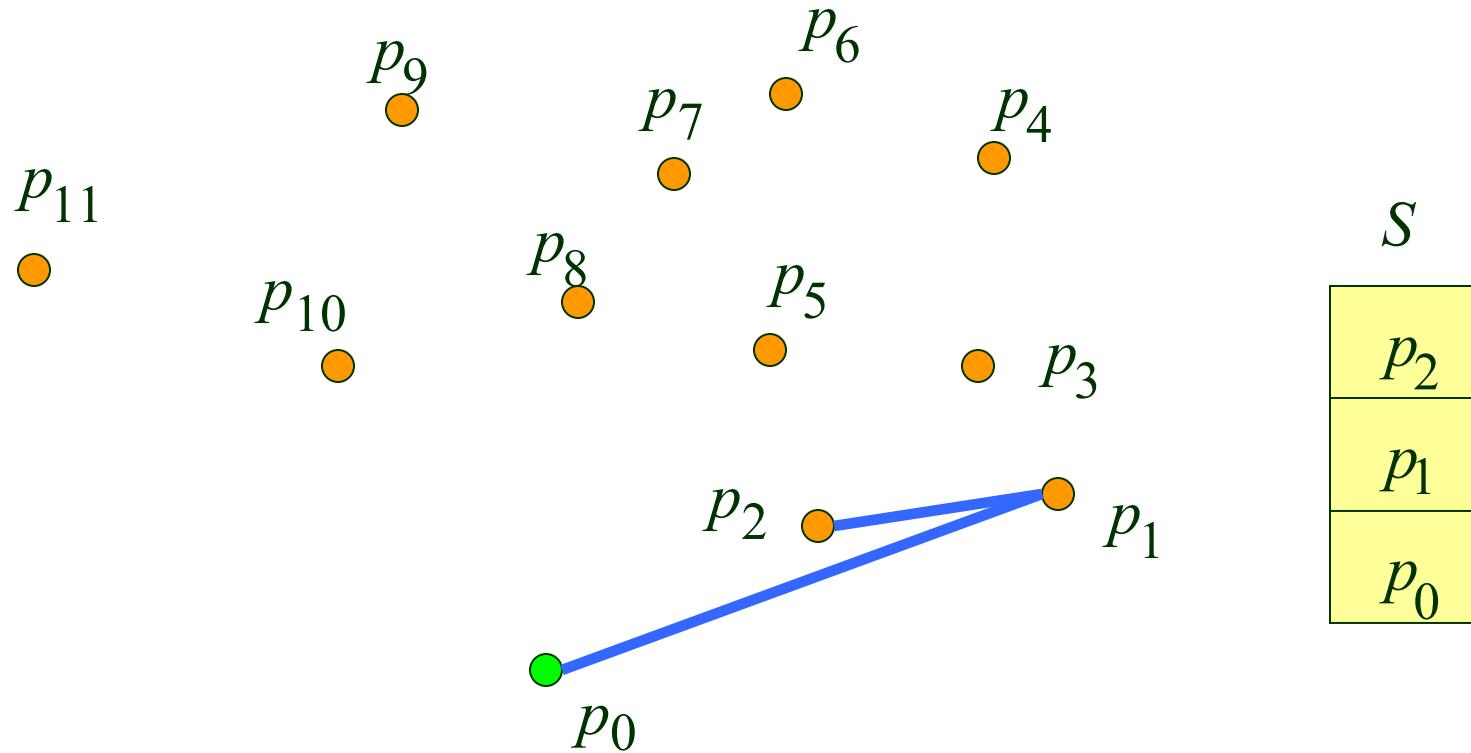


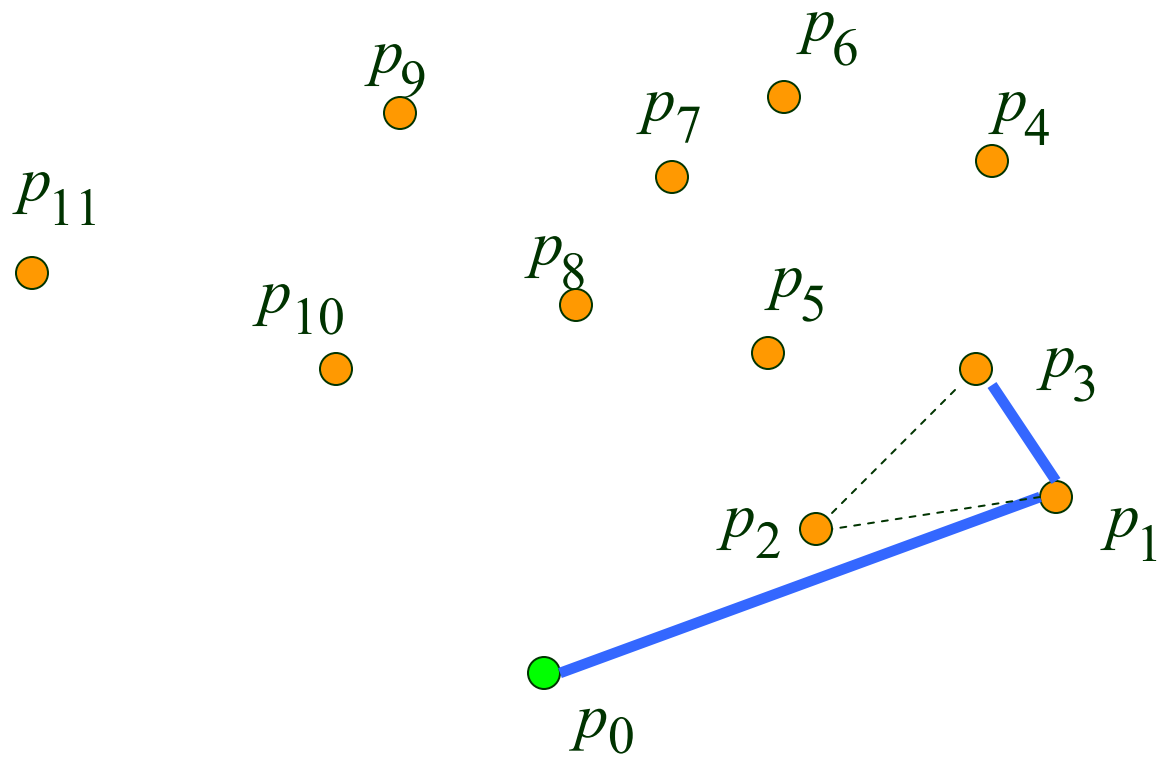
How to break a tie?

Labels are in the polar angle order.

(What if two points have the same polar angle?)

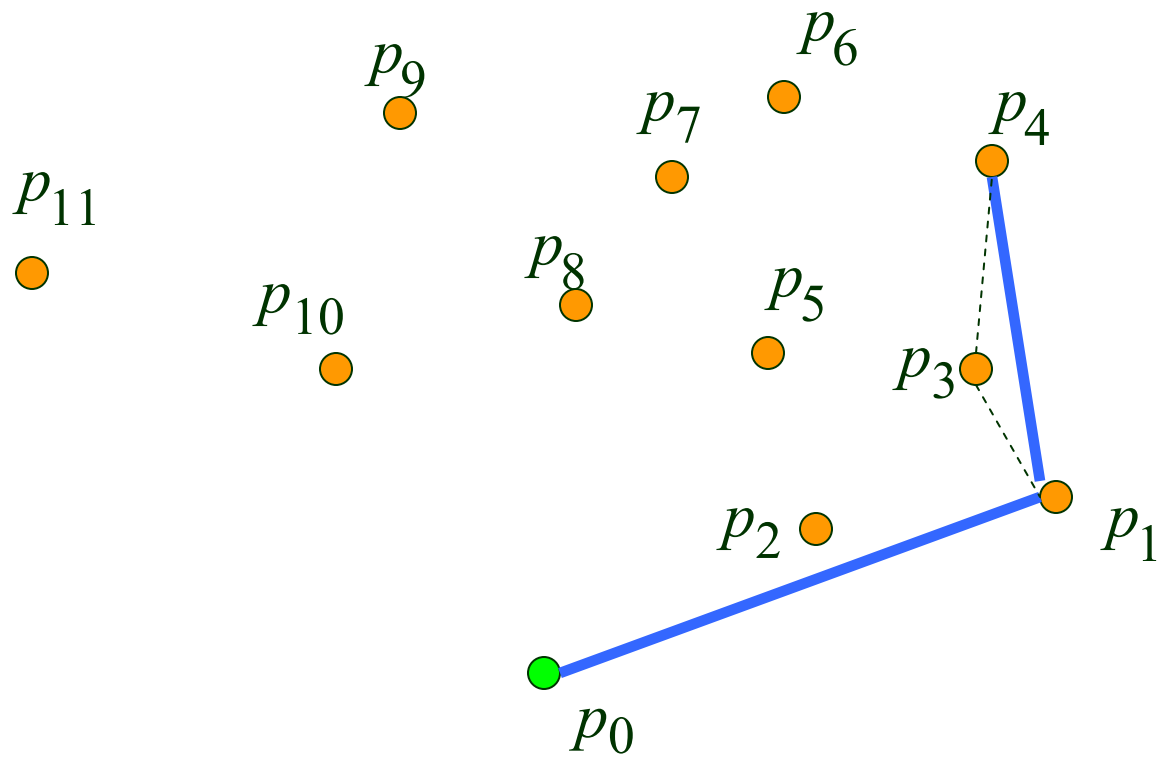
Stack Initialization





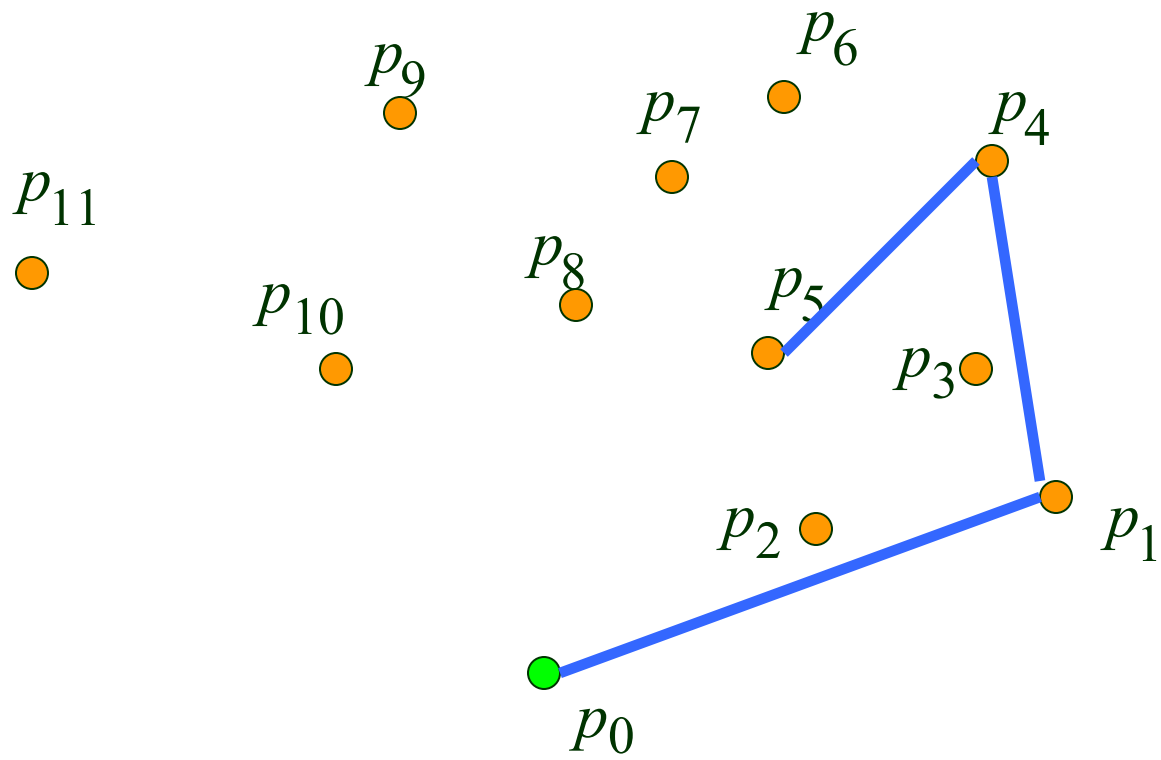
S

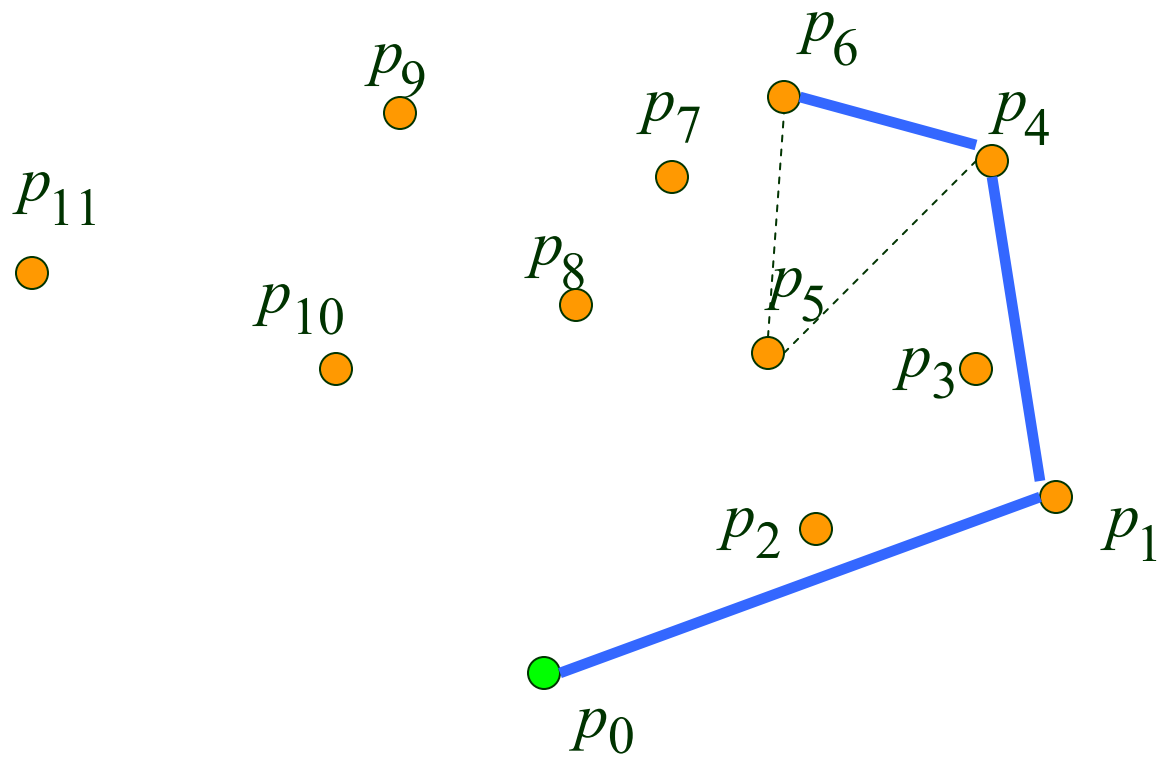
p_3
p_1
p_0



S

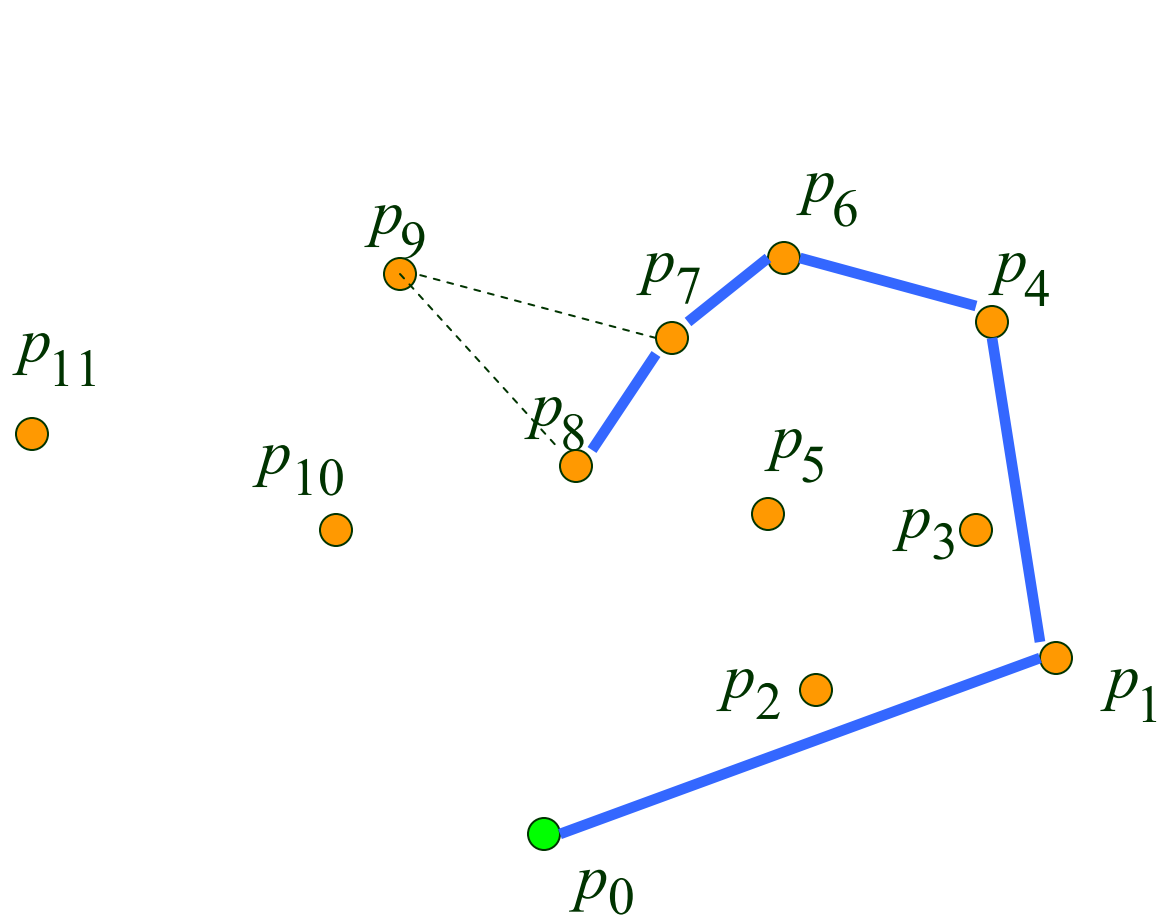
p_4
p_1
p_0





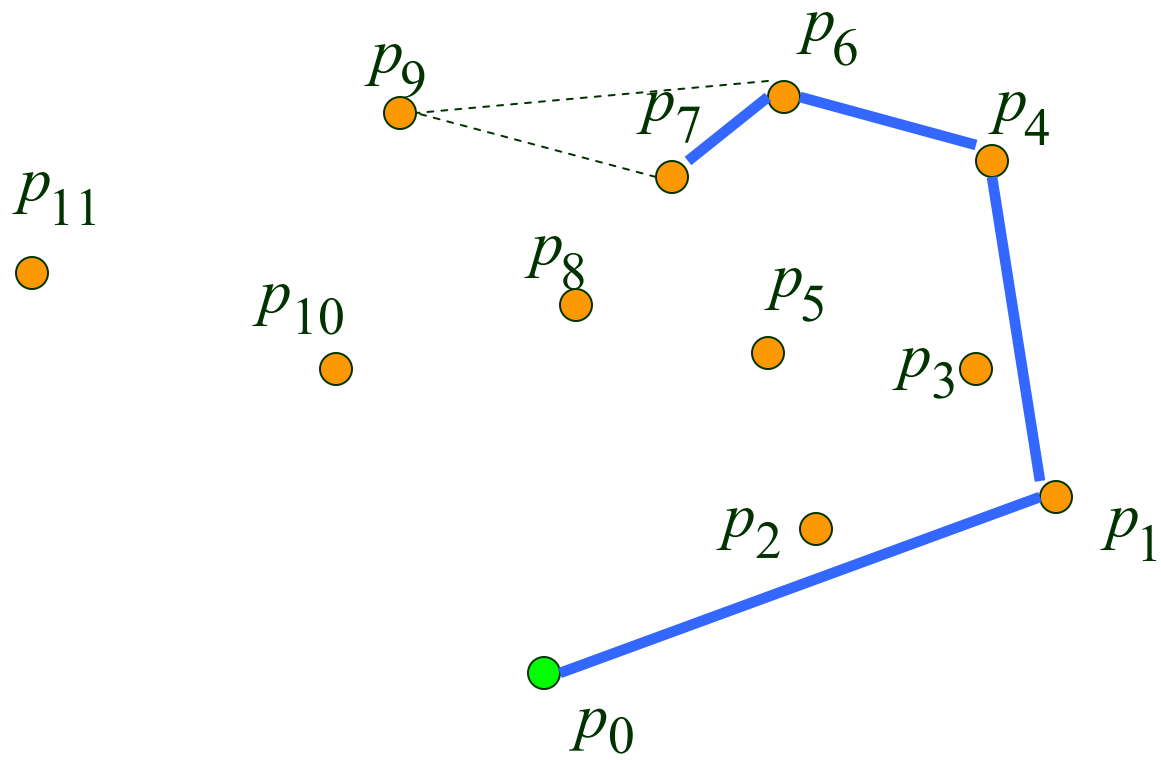
S

p_6
p_4
p_1
p_0



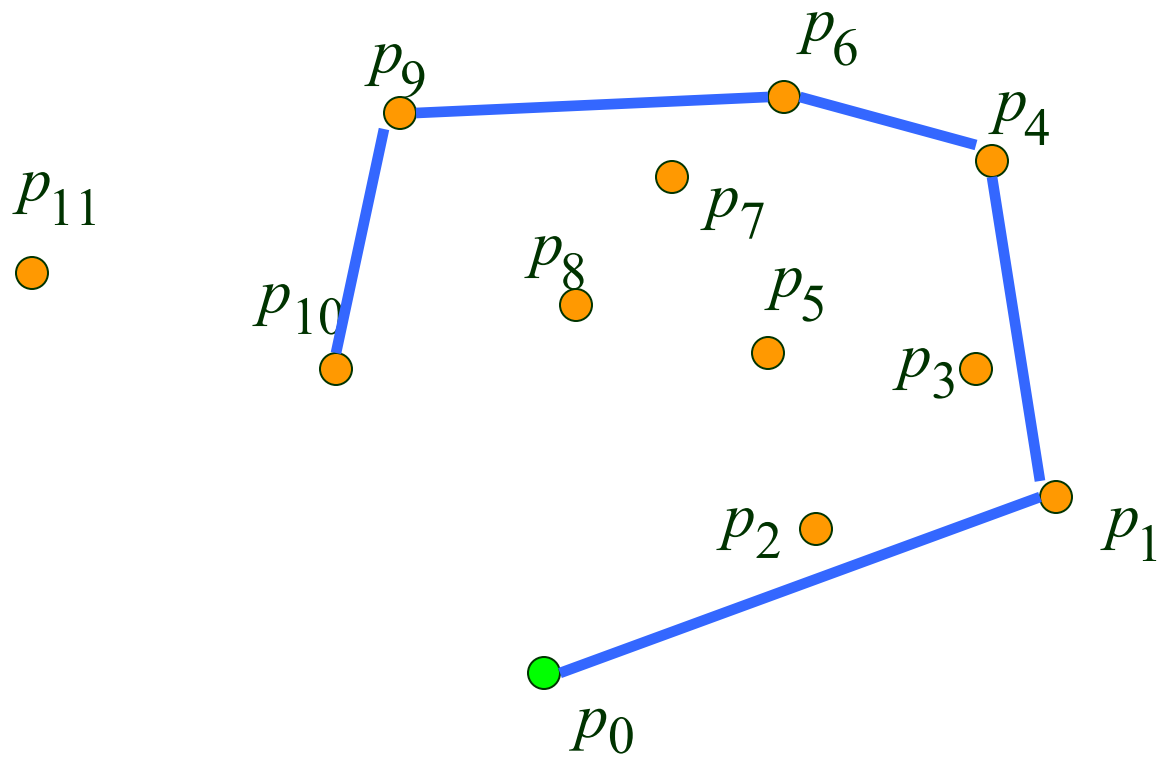
S

p_8
p_7
p_6
p_4
p_1
p_0



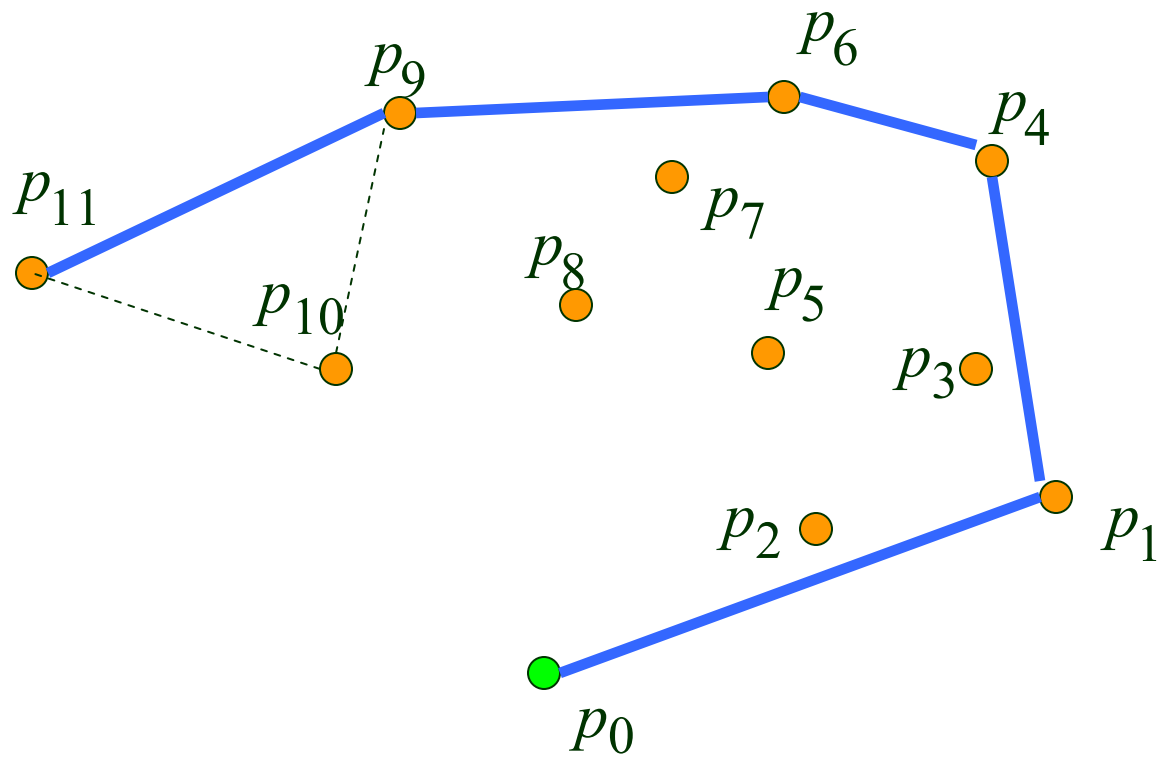
S

p_7
p_6
p_4
p_1
p_0



S

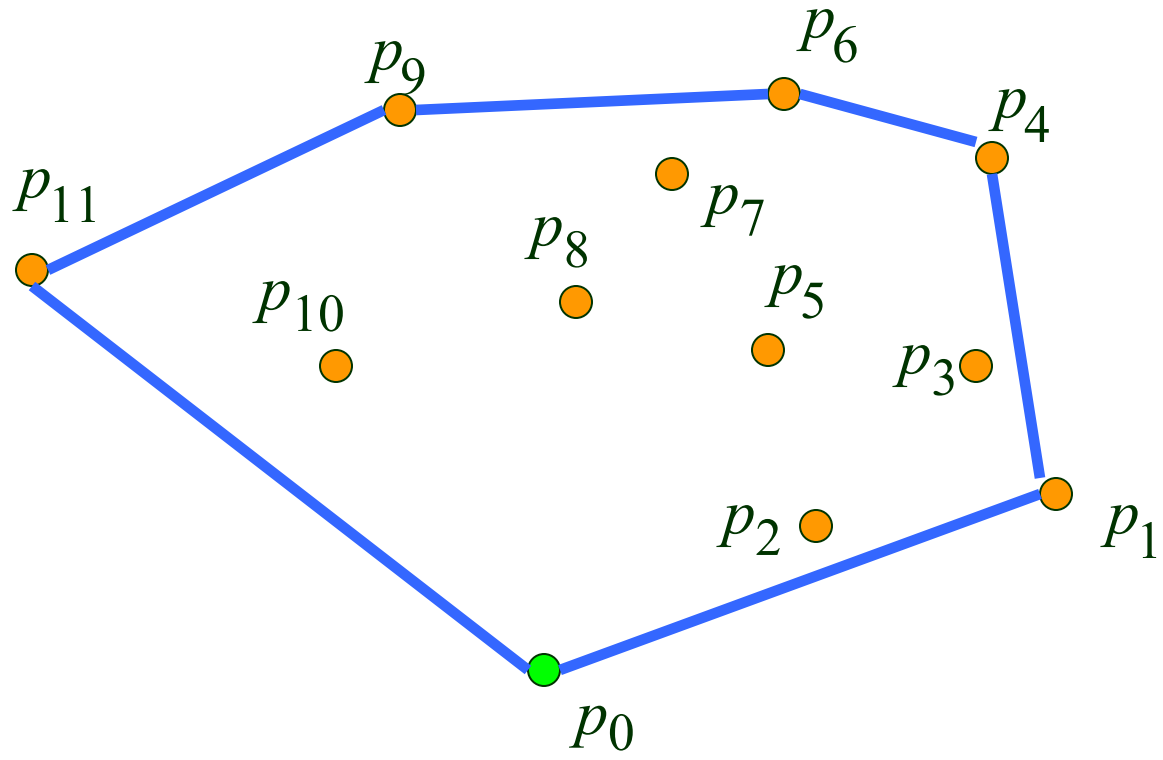
p_{10}
p_9
p_6
p_4
p_1
p_0



S

p_{11}
p_9
p_6
p_4
p_1
p_0

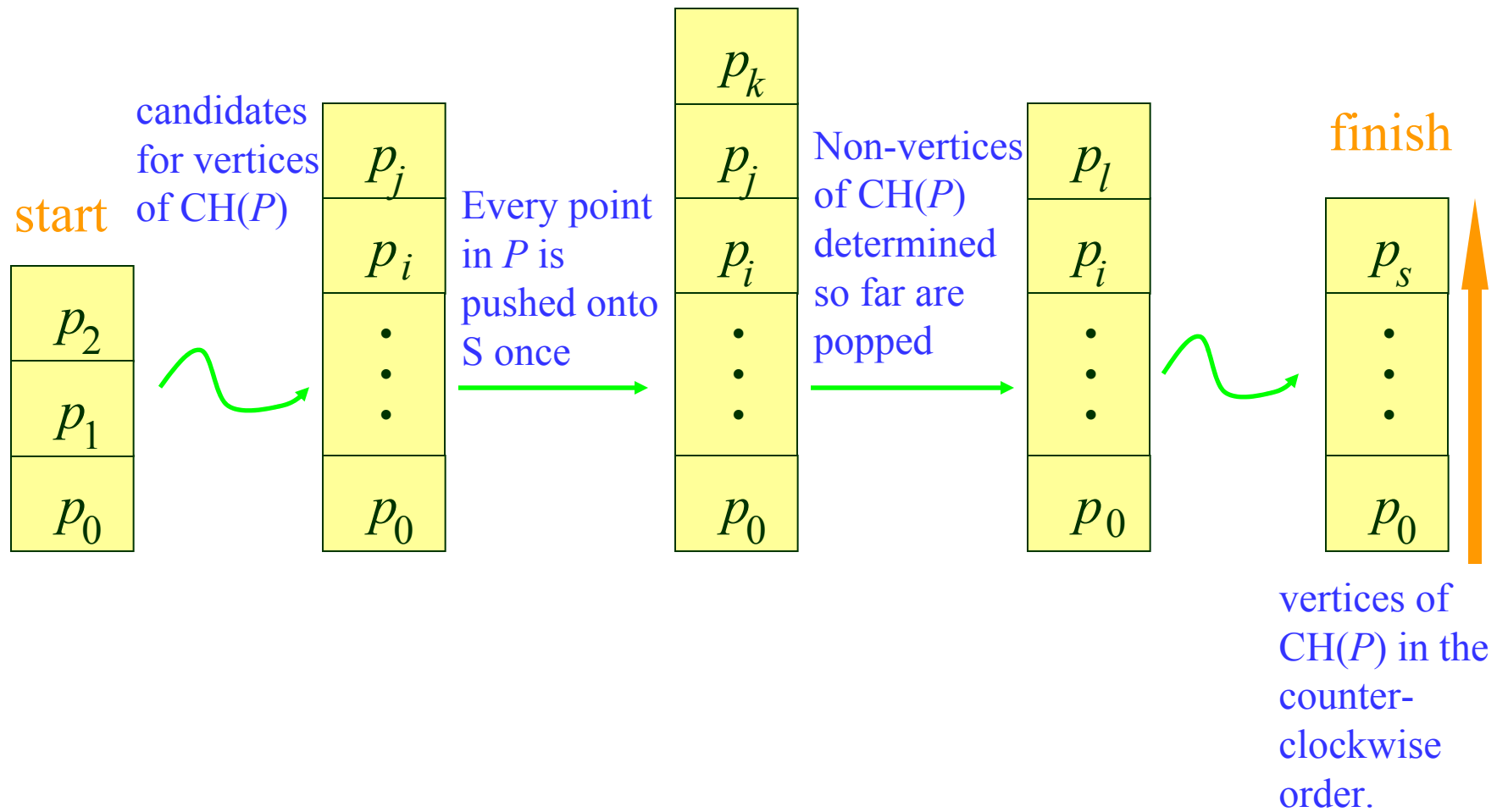
Finish



S

p_{11}
p_9
p_6
p_4
p_1
p_0

Graham's Scan



The Graham Scan Algorithm

Graham-Scan(P)

let p_0 be the point in P with *minimum* y -coordinate

let $\langle p_1, p_2, \dots, p_{n-1} \rangle$ be the remaining points in P

sorted in counterclockwise order by polar angle around p_0 .

Top[S] $\leftarrow 0$

Push(p_0, S)

Push(p_1, S)

Push(p_2, S)

for $i \leftarrow 3$ to $n - 1$

do while p_i makes a *nonleft* turn from the line segment
determined by Top(S) and Next-to-Top(S)

do Pop(S)

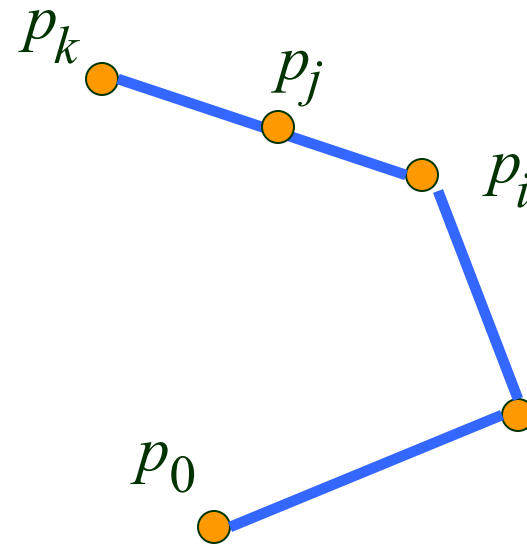
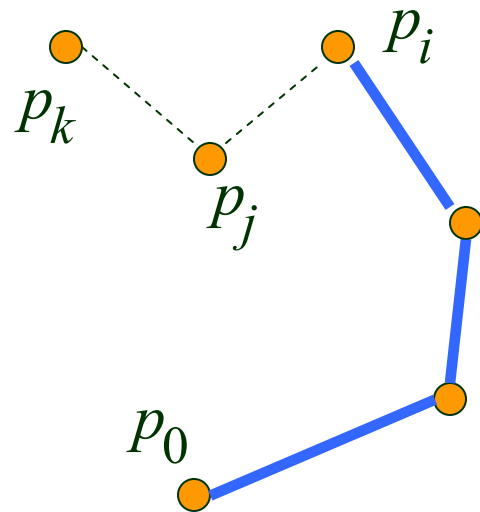
Push(S, p_i)

return S

Proof of Correctness

Claim 1 Each point popped from stack S is not a vertex of $\text{CH}(P)$.

Proof Two cases when p_j is popped:



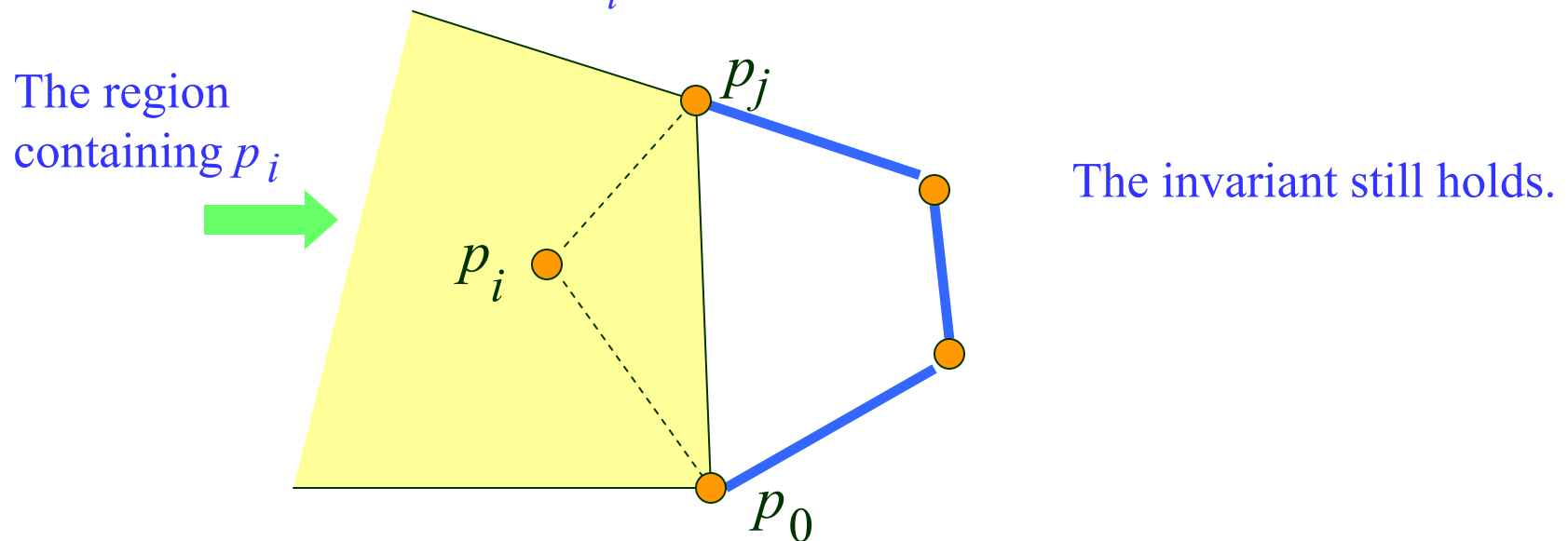
In neither case can p_j become a vertex of $\text{CH}(P)$.

Claim 2 Graham-Scan maintains the *invariant* that the points on stack S always form the vertices of a convex polygon in counterclockwise order.

Proof ✨ The claim holds right after initialization of S when p_0, p_1, p_2 form a triangle (which is obviously convex).

✨ Popping a point from S preserves the invariant.

✨ Consider a point p_i being pushed onto S .



Correctness of Graham's Scan

Theorem If Graham-Scan is run on a set P of at least three points, then a point of P is on the stack S at termination if and only if it is a vertex of $\text{CH}(P)$.

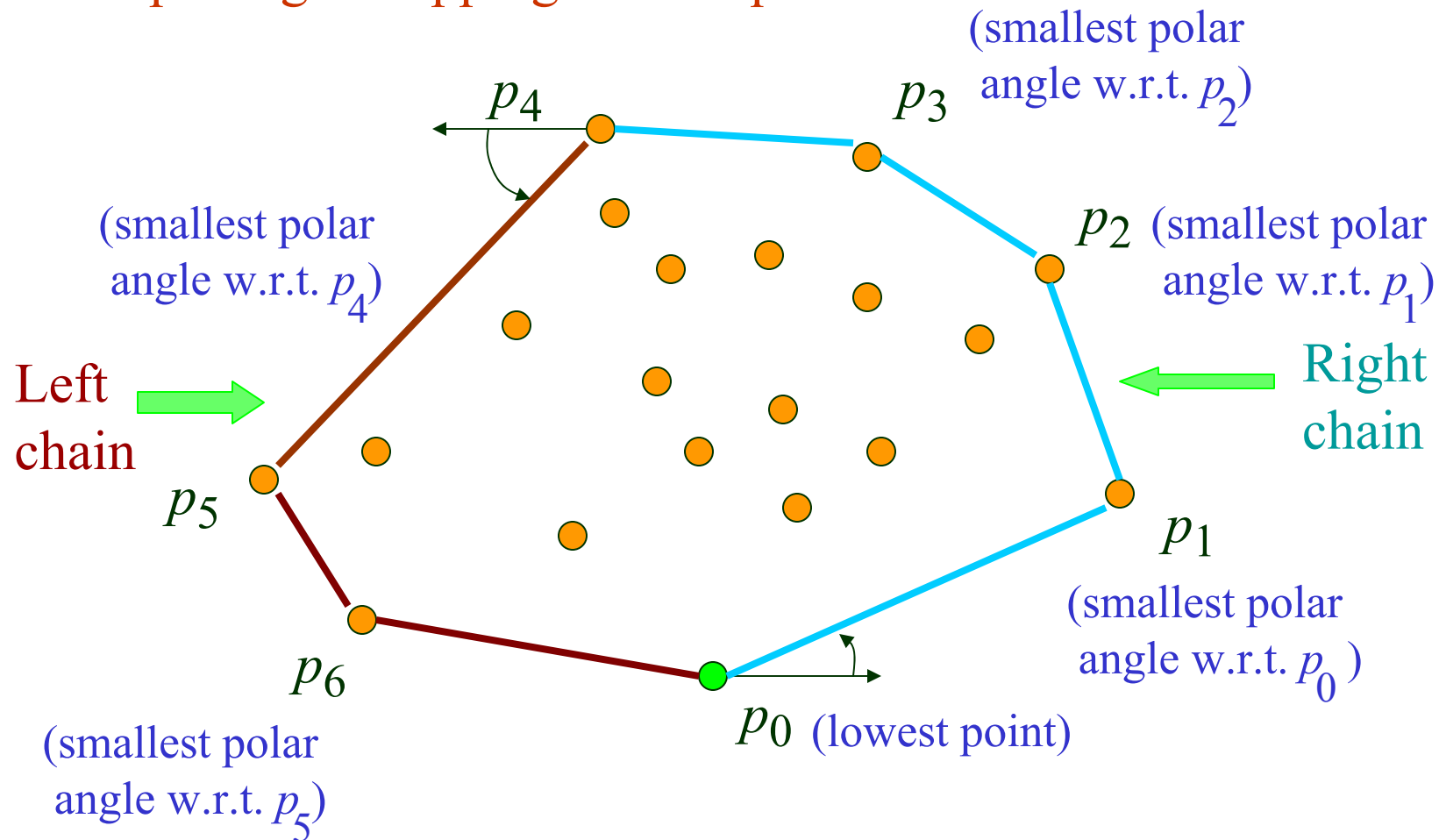
Running time

	#operations	time / operation	total
Finding p_0	1	$\Theta(n)$	$\Theta(n)$
Sorting	1	$O(n \lg n)$	$O(n \lg n)$
Push	n	$O(1)$	$\Theta(n)$
Pop	$\leq n - 2$ Why?	$O(1)$	$O(n)$

The running time of Graham's Scan is $O(n \lg n)$.

Jarvis' March

A “package wrapping” technique



Running Time of Jarvis's March

Let h be the number of vertices of the convex hull.



For each vertex, finding the point with the minimum Polar angle, that is, the next vertex, takes time $O(n)$.

Comparison between two polar angles can be done using cross product.

Thus $O(nh)$ time in total.