Variable Assignent in Modality

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I

An incredible theorem of modal predicate logic named $\Box$I has caused considerable controversy:

$$\Box\mathrm{I} \quad x = y \supset \Box x = y$$

Here is a simple proof:

$$\begin{array}{ll}
(1) x = y \supset (\Box x = x \supset \Box x = y) & \mathrm{I2} \\
(2) \Box x = x \supset (x = y \supset \Box x = y) & (1) \times \mathrm{PC} \\
(3) \Box x = x & \mathrm{I1} \times \mathrm{N} \\
(4) x = y \supset \Box x = y & (2)(3) \times \mathrm{MP}
\end{array}$$

However, there seem to be apparent counterexamples, such as

Hesperus = Phosphorus

The number of planets = 8

Water = H_2O

The first Postmaster General = the inventor of bifocals

One would expect that all of them are *contingent*, rather than *necessary*.

There are many analyses and theories about this, including definite description and reference. But now let us investigate it directly from the point of view of modal logic, and see what happens.

In terms of validity, $\Box$I says that in every model $\langle W, R, D, V\rangle$, for any world $w \in W$ and any variable assignment μ, $V_\mu(x = y \supset \Box x = y, w) = 1$. Assume $V_\mu(x = y, w) = 1$, then $V_\mu(\Box x = y, w)$ should be 1, which means $V_\mu(x = y, w') = 1$ for all w' that w can see. This is indeed the case, since $\mu(x) = \mu(y)$. So basically $\Box$I tells us that if $x = y$, then go to other possible worlds *with fixed variable assignment*, x still equals to y. It sounds trivial, and different from the way of thinking. There should be possible worlds in which the variable assignment may change. In other words, changing the variable assignment is a kind of possibility! The number of planets is 8 in the actual world, but in another possible world it could be 9. In the actual world, Hesperus is that planet (say Venus, if you can regard Venus as an essential object. Or you may use "that" to represent it, like Russell), and Phosphorus is the same planet. But in another possible world, Hesperus could be another planet, so as Phosphorus, and of course they could be different. The same analysis applies to "water = H_2O" and "the first Postmaster General = the inventor of bifocals". Here we regard names as *assigned variables*, e.g. "Hesperus", "Phosphorus", "water", "the number of planets", "the first Postmaster General", and "the inventor of bifocals". And essential elements (they are basic, or abstract with some certain property) are objects in the domain, e.g. "9", "H", "O", "H_2O", "that person", "that planet". We will come back to this later, in section IV.

The key problem is the following rule in semantics for modal LPC (Lower Predicate Calculus):

$$[V\Box] \qquad V_\mu(\Box\alpha, w) = 1 \text{ iff } V_\mu(\alpha, w') = 1 \text{ for every } w' \text{ such that } wRw'.$$

Given assignment μ, it is necessary that α is true iff α is true in all possible worlds with respect to the same assignment μ. But why do we stick to the same assignment? If it is *really necessary* that α is true, and we can imagine possible worlds with different assignments to assigned variables in the actual world, why do we stick to the same assignment?

II

One way to resolve this problem is to make variable assignment world-relative. Instead of $\mu(x)$, we have $\mu(x, w)$, and the rule for evaluating atomic wffs becomes:

$$[V\phi'] \qquad V_\mu(\phi(x_1, \cdots, x_n), w) = 1 \text{ iff } \langle \mu(x_1, w), \cdots, \mu(x_n, w), w\rangle \in V(\phi).$$

All the other rules remain as before except that in $[V\forall]$, we need to generalize the notion of an x-alternative so that ρ is an x-alternative of μ iff for every variable y except x, and every $w \in W$, $\rho(y, w) = \mu(y, w)$.

This semantics does not verify $\Box$I. And this system is called contingent identity system. However, it is noteworthy that the variable assignment is still fixed, just with more choices. That is to say, μ is determined before $\Box$, which is undesirable. It is not the way of thinking that one should imagine a world-relative assignment first, then go to other possible worlds following that assignment. Instead, when we say something is necessary, we are free to *change* the variable assignment, and it should be true all the way.

Therefore, the most natural way is to modify $[V\Box]$ to be:

> $[V\Box']$ $V_\mu(\Box\alpha, w) = 1$ iff $V_{\mu'}(\alpha, w') = 1$ for every w' such that wRw', and for every μ'.

All the other rules remain the same as before. Now our variable assignment is not *world-relative*, but *world-changeable*. And because of the meaning of necessity, it can change to any possible value.

Note that for $V_\mu(\alpha, w)$, μ only affects free variables in α. Hence as long as $\Box\alpha$ is a de dicto claim, our new rule has exactly the same effect as the original one. If you are an antiessentialist, like Quine, do not think de re claims make any sense, then everything is perfect. So how about de re claims? Under our new rule, when will a de re claim, e.g. $\exists x\Box\phi x$, be true? Fix any w', $V_{\mu'}(\alpha, w') = 1$ for every μ' is equivalent to $V(\forall x_1 \cdots \forall x_n \alpha, w') = 1$, where $\{x_1, \cdots, x_n\}$ is the set of all free variables in α. Since the latter does not depend on variable assignment, we omit the subscript here. Thus, we have the following fact:

$$\Box\alpha \equiv \Box\forall x_1 \cdots \forall x_n \alpha, \qquad x_1 \cdots x_n \text{ are all free variables in } \alpha$$

Then $\exists x\Box\phi x$ is equivalent to $\Box\forall x\phi x$, the quantifier $\exists x$ cannot quantify into the $\Box$. At first glance, it seems like a surprising result. Nonetheless, it is reasonable and is what we want. A claim like “the inventor of bifocals necessarily invents bifocals” is false. Namely, Benjamin Franklin does not necessarily invent bifocals. How could this be true? Well, if it is necessary that everyone invents bifocals, then someone necessarily invents bifocals! So a claim like $\forall x\Box(\phi x \supset \phi x)$ is true under our new rule, as what is supposed to be.

Similarly, when we say “it is possible that α”, once α is true in a possible world with some variable assignment, it is true, e.g. “it is possible that the number of planets = 9”, “it is possible that the first Postmaster General = the inventor of electric light”. Hence we have:

> $[V\Diamond']$ $V_\mu(\Diamond\alpha, w) = 1$ iff $V_{\mu'}(\alpha, w') = 1$ for some w' such that wRw', and for some μ'.

corresponding to $[V\Box']$. Again, the effect is on free variables in α. Specifically,

$$\Diamond\alpha \equiv \Diamond\exists x_1 \cdots \exists x_n \alpha, \qquad x_1 \cdots x_n \text{ are all free variables in } \alpha$$

Since

$$\begin{aligned} \Box\alpha &\equiv \Box\forall x_1 \cdots \forall x_n \alpha && [V\Box'] \\ &\equiv \neg\Diamond\neg\forall x_1 \cdots \forall x_n \alpha && \text{LMI} \\ &\equiv \neg\Diamond\exists x_1 \cdots \exists x_n \neg\alpha && \text{QI} \\ &\equiv \neg\Diamond\neg\alpha && [V\Diamond'] \end{aligned}$$

Under our new rules $[V\Box']$ and $[V\Diamond']$, LMI still holds:

$$\Box\alpha \equiv \neg\Diamond\neg\alpha$$

III

We know that validating the Converse Carban Formula is a serious disadvantage of world-relative variable assignments.

Converse Carban Formula $\qquad \Box\exists x\alpha \supset \exists x\Box\alpha$

The Converse Carban Formula leads to ridiculous conclusions. In a competition, it is necessary that someone will win, but there is no one of whom it is necessary that he/she will win. It is necessary that some number is the number of planets (say 8, 9, or any other number), but there is no number such that it is necessary that the number of planets is that number.

One may argue that the Converse Carban Formula holds for intensional objects, say “the winner”. Whereas, under our rule $[V\Box']$, we need not to take such troubles—the Converse Carban Formula is certainly invalid.

For simplicity, suppose x is the only free variable in α, we have

$$\exists x\Box\alpha \equiv \Box\forall x\alpha$$

which is stronger than $\Box\exists x\alpha$. The former is weaker than the latter, and hence the Converse Carban Formula is not valid. Here is a simple counterexample for instance $\Box\exists x\phi x \supset \exists x\Box\phi x$: let $W = \{w_1, w_2\}$, $R = W \times W$ (every world can see all the worlds), $D = \{u_1, u_2\}$, $V(\phi) = \{\langle u_1, w_1\rangle, \langle u_2, w_2\rangle\}$. Then $\Box\exists x\phi x$ is valid in this model, but $\Box\forall x\phi x$ is not valid, i.e. $\exists x\Box\phi x$ is not valid.

IV

Sometimes we do want fixed variable assignments. We would like to believe that "9=4+5" is a necessary identity, not just a contingent identity. Variables with fixed assignment are constants. And theories of individual constants and function symbols can be introduced into our system, exactly the same way as before. We can have *terms*, or eliminate them using Russell's theory of descriptions. Individual constants are essentially equivalent to the object they refer to, and that is why I regard them as objects (cf. section I).

Consider the question "who could have been the U.S. president?". Imagine that the domain consists of all the people. (There are other objects, but they are irrelevant to our question and we are not concerned about them.) We have an assigned variable "the U.S. president", and μ(the U.S. president) = Obama. But it is not necessary that "the U.S. president = Obama". More precisely, we should give a name for object Obama, say $V(o) =$ Obama, and the proposition becomes "the U.S. president = o". Or by Russell's theory of descriptions, using predicate ϕ_o such that $\langle u, w\rangle \in V(\phi_o)$ iff $u =$ Obama, the proposition becomes "the U.S. president = $(\iota x : \phi_o x)$". But as mentioned before, individual constants and objects are essentially equivalent, and hence we just abuse the notation by saying "the U.S. president = Obama" for convenience. Analogous situations include: "μ(the number of planents) = 8" and proposition "the number of planents = 8", "μ(water) = H_2O" and proposition "water = H_2O", "μ(Hesperus) = μ(Phosphorus) = Venus" and proposition "Hesperus = Phosphorus".

Back to our question,

$$\text{the U.S. president} = \text{Obama} \supset \Box \text{ the U.S. president} = \text{Obama} \qquad (*)$$

is false, because the former is true when μ(the U.S. president) = Obama, but we encounter $\Box$, then μ(the U.S. president) can change to anyone and is no longer bound to Obama.

In fact, between individual variables and individual constants, we can have things in the middle. For instance, we may want the claim

$$\Box\text{American(the U.S. president)} \qquad (**)$$

to be true, where "American" is a predicate. How can we do that? We can restrict the change of the variable assignment. In this model, let us restrict $[V\Box']$ to be:

$[V\Box']_1$ $V_\mu(\Box\alpha, w) = 1$ iff $V_{\mu'}(\alpha, w') = 1$ for every w' such that wRw', and for every μ' such that $\mu'(x) \in \{\text{American}\}$ for all x.

where {American} is a subset of domain {people}. Now (∗∗) becomes true, and (∗) remains false.

The above rule is somewhat specific. Let us try to generalize it. Imagine that we have all kinds of objects: humans, birds, flowers, rocks, etc. If the value of an assigned variable is a human, e.g. "the inventor of bifocals", and we want it to be able to be any other person, but we don't want it to be a non-human object, we can do this:

$[V\Box']_2$ $V_\mu(\Box\alpha, w) = 1$ iff $V_{\mu'}(\alpha, w') = 1$ for every w' such that wRw', and for every μ' such that if $\mu(x) \in \{\text{humans}\}$, then $\mu'(x) \in \{\text{humans}\}$.

Perhaps you appreciate counterfactuals like "if I were a bird", and thus you dislike the above rule. You may want:

$[V\Box']_3$ $V_\mu(\Box\alpha, w) = 1$ iff $V_{\mu'}(\alpha, w') = 1$ for every w' such that wRw', and for every μ' such that if $\mu(x) \in \{\text{animals}\}$, then $\mu'(x) \in \{\text{animals}\}$.

More generally, it can be the following form:

$[V\Box']_4$ $V_\mu(\Box\alpha, w) = 1$ iff $V_{\mu'}(\alpha, w') = 1$ for every w' such that wRw', and for every μ' such that for all x, $\mu(x)$ and $\mu'(x)$ belong to the same class.

where the class depends on one's choice. Note that original $[V\Box]$ is a special case of $[V\Box']$, in which $\mu'(x) = \mu(x)$ is required. In a word, $[V\Box']$ is very robust.

The last comment I would like to make is that world-changeable variable assignments are naturally suitable for changing domains. You can restrict μ' to fall in $D_{w'}$ or D_w. Correspondingly, you validate the Barcan Formula or not in expanding domains. Likewise, you validate the Converse Barcan Formula or not in shrinking domains.

V

A remark of contingent identity systems with $[V\phi']$ is that it does not validate all instances of Leibniz's axiom,

Leibniz's axiom $x = y \supset (\alpha \supset \beta)$ – α and β differ only in that α has free x in 0 or more places where β has free y.

but it does validate all instances in which α and β contain no modal operators.

Our $[V\Box']$ and $[V\Diamond']$ rules also have this property. Furthermore, we can make a parallel remark that all de dicto claims are intact, and de re claims are converted into de dicto form in the way of thinking.

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