COMPUTER VISION, GRAPHICS, AND IMAGE PROCESSING 33, 116–125 (1986)

NOTE

A Note on the Gradient of a Multi-Image

SILVANO DI ZENZO

IBM Rome Scientific Center, Via del Giorgione 129, Rome, Italy

Received December 1, 1984; accepted June 14, 1985

Gradient based edge detection techniques can be extended to multispectral images in various ways: difference operators can be applied to each component of a multi-image, and the results can be combined, e.g., taking the RMS, or the sum, or the maximum of their absolute values. In all of these approaches the image-components do not actually cooperate with one another, i.e., edge evidence along a given direction in one component does not reinforce edge evidence along the same direction in other components. To avoid this, the use of the tensor gradient of multi-images regarded as vector fields is suggested. Explicit formulas for the direction along which the rate of change is maximum, as well as for the maximum rate of change itself, are derived. Digital approximations are obtained by surface fitting. © 1986 Academic Press, Inc.

INTRODUCTION

Edge detection is of interest for both image analysis and image understanding. In image segmentation, certain specific parts of an image (objects or regions) are logically extracted, i.e., separated from the rest of the image, based on some local property of the image, usually the gray level. Object extraction is often a prerequisite for object description or recognition. For example, in character recognition, the characters must be extracted before their recognition is attempted; in chromosome classification, the chromosomes must first be extracted from the photomicrograph before they can be classified. Object extraction may be critical when there are neighboring objects which do not differ significantly in any pixel property.

In many cases, edge detection turns out to be a powerful tool for object extraction: indeed, an efficient way of extracting an object in an image is often to detect and reconstruct its border. Edge detection has received considerable attention in recent years. A systematic account of edge detection can be found in Rosenfeld and Kak [14]. The literature on edge detection has been surveyed by Davis [2], Fu and Mui [4], Peli and Malah [8], and Di Zenzo [3]. Edge detection is of interest also in artificial intelligence as it is efficiently implemented in biological visual systems. Edge detection from this viewpoint has been reviewed by Brady [1].

Currently, a number of well established techniques for edge detection in ordinary (one-band) images are available. In many cases, edges are detected by applying some local operator (usually some digital approximation of the gradient) and then thinning and linking short edges to form whole boundaries. The use of the gradient dates back to Kovasznay and Joseph [5, 6]. A well-known digital approximation of the gradient is that proposed by Roberts [11]. The components of the Roberts gradient are obtained by convolving the image with

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

0734-189X/86 $3.00
Copyright © 1986 by Academic Press, Inc.
All rights of reproduction in any form reserved.

A digital gradient magnitude is then obtained as the maximum of the absolute values of the components.

More effective digital approximations of the gradient can be obtained by computing differences of local averages (which is equivalent to smoothing the picture before applying the gradient operator). For example, a digital approximation of the derivative $\partial f/\partial x$ of an image $f(x, y)$ can be obtained by convolving f with

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Differences of averages produce thick bands of edge data even when the edge is an ideal step edge. In real images, however, edges are also always diffused and noisy, and a certain thickness of the detected edge band is unavoidable, as it is shown by an analysis of the edge detection process in the frequency domain (see, e.g., [14]).

The use of operators based on differences of averages raises a problem about the size of the neighborhoods on which the averages are to be computed. In general, different sizes are needed at different points. A classic solution to this problem is that proposed by Rosenfeld and Thurston [12] using a set of images averaged over larger and larger neighborhoods.

Differences of weighted averages have also been proposed. An example is the Sobel operator, whose x and y components are the convolutions of the given image with

$$\frac{1}{4}\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

Surface fitting is another classical method of deriving digital approximations for the gradient (and the Laplacian) of a digital image. A digital image can be approximated locally at any given point by a polynomial function, e.g., a plane. The gradient and the Laplacian of the polynomial function can then be taken as digital approximations for the gradient and the Laplacian of the image at the given point. Many well-known digital approximations of different operators can be regained in this way, as discussed by Prewitt [9]. For example, least squares fitting a plane $z = ax + by + c$ to the gray levels in a 2×2 neighborhood leads to the approximations to the first derivatives along x and y obtained as the convolutions of the given image with

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The least squares fitting of a second degree polynomial to the gray levels in a 3×3 neighborhood gives the following approximations to the first derivatives along x and y, respectively:

$$\frac{1}{3}\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

which are known as Prewitt operators. More recently, Morgenthaler and Rosenfeld [17] derived digital approximations for the gradient in nD images.

We have reviewed these notions on surface fitting as we shall make use of them to obtain digital approximations for the differential operators we derive below. To derive digital approximations of the gradient is just one of the techniques which can be used to obtain edge detectors. Other techniques include mask matching, step fitting, use of decision theory, and filtering in the frequency domain.

The subject of the present article is gradient-based edge detection in multi-images. Research on this subject has already been done by Nevatia [18], Robinson [19], and Sankar [20], among others. We shall discuss difference operators that make use of vectors of local properties rather than gray levels. All the techniques described above can be extended to multi-images in various ways. Our purpose is to discuss the type of extensions suggested by the analytical treatment of multi-images as vector fields. This approach has been suggested by Machuca and Phillips [7] who established a framework for the applications of differential geometry to image processing. The present article proposes a solution for a problem formulated by these authors, namely "how to combine the gradients (of the image components) into one output." The analytical treatment begins with the next section; in the rest of this section, a brief qualitative discussion of the problem is provided to place it in better focus.

As a first approximation, all the techniques described above can be extended to multi-images. Since a multi-image can be modelled as an array of ordinary images, a straightforward approach is to evaluate the gradient as the vector sum of the gradients of the individual components of the multi-image. Another approach is to take the RMS of the component gradient magnitudes as the magnitude of the resultant gradient. For example, if R, G, and B are the respective red, green, and blue components of a color image, the RMS of $\nabla_x R$, $\nabla_x G$, and $\nabla_x B$

$$\left\{(R(x, y) - R(x-1, y))^2 + (G(x, y) - G(x-1, y))^2 + (B(x, y) - B(x-1, y))^2\right\}^{1/2}$$

could be taken as the x-component of the resultant gradient. The resultant gradient magnitude would turn out to be the Euclidean distance between the color vectors $(R(x-1, y)\ G(x-1, y), B(x-1, y))$ and $(R(x, y), G(x, y), B(x, y))$. As color vectors (or, more generally, property vectors) can be averaged over neighborhoods, the distance between point vector values can be replaced by a distance between averaged vector values.

The RMS can be replaced by the sum, or even the maximum, of the absolute values of the differences involved. The Fisher distance can also be used at the price of some additional computations (both averages and variances are to be computed over suitable neighborhoods).

The above mentioned operators make use of color (or spectral) components; they, however, can be unsatisfactory in certain cases. For example, the use of Fisher distance can provide good results, but is time consuming. On the other hand, taking the vector sum of the gradients of the separate bands as the gradient of the whole multi-image is computationally cheap but very unsatisfactory. Consider, for example, a color image where B is constant while R and G both show vertical edges around, say, the y axis. Suppose that the edge strength is the same but one band increases from left to right while the other decreases: then the vector sum of the gradients would provide a null resultant gradient. The RMS approach would provide the correct result in the above case.

Consider, however, the following two cases: at any given point (x, y) all the bands exhibit an edge (a) along one and the same direction, (b) along different, possibly orthogonal, directions; with equal band edge strengths, the RMS approach gives the same result in both cases, while, quite obviously, in case (a) the edge strength should be greater as convergent evidence from all the bands should reinforce the edge strength.

THE GRADIENT OF A MULTI-IMAGE

In what follows, R denotes the set of real numbers. Let f: $R^2 \to R^m$ be a continuous multi-image. We set $V_2 = \{f(x): x \in R^2\}$. The following notations will be adopted: $x = (x^1, x^2), f = (f^1, \ldots, f^m)$, $y = f(x) = (f^1(x), \ldots, f^m(x)) = (y^1, \ldots, y^m)$. Hence, for $j = 1, \ldots, m$, $y^j = f^j(x)$.

We assume the Jacobian $|\partial f^j / \partial x^h|$ to be of rank 2 everywhere in R^2. Then V_2 is a two-dimensional manifold embedded in R^m. Let $h = 1, 2$ and $f_h(x) = (\partial f^1/\partial x^h, \ldots, \partial f^m/\partial x^h)$. So defined, $f_h(x)$ is an m tuple of reals in which the jth term represents the value of $\partial f^j/\partial x^h$ at $x \in R^2$. We assume that $f_h(x)$ and its first derivatives are continuous.

Let $\times$ denote the scalar product in R^m. For $h, k = 1, 2$, we set

$$g_{hk}(x) = f_h(x) \times f_k(x). \tag{1}$$

Notice that $\{f_h(x)\}_{h=1,2}$ is a basis for the two-dimensional vector space of tangent vectors of V_2 at $y = f(x)$. The four numbers $g_{hk}(x)$, h, $k = 1, 2$ are the components of a symmetric tensor field $g(x) = g(x^1, x^2)$ of rank 2.

For image processing applications, we are interested in the following two quantities to be computed locally at each point $x = (x^1, x^2)$: (a) the direction through point x along which f has the maximum rate of change, and (b) the absolute value of this maximum rate of change. We are thus led to the problem of maximizing the form

$$df^2 = \sum_{h=1}^{2} \sum_{k=1}^{2} g_{hk}\, dx^h\, dx^k \tag{2}$$

under the condition

$$\sum_{h=1}^{2} dx^h\, dx^h = 1. \tag{3}$$

This problem can be reformulated as the problem of finding that value of θ which maximizes the form

$$F(\theta) = g_{11}\cos^2\theta + 2g_{12}\cos\theta\sin\theta + g_{22}\sin^2\theta \tag{4}$$

which, in turn, can be solved by means of the following substitutions

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$
$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta.$$

One obtains

$$\begin{aligned} F(\theta) &= \{g_{11}(1 + \cos 2\theta) + 2g_{12}\sin 2\theta + g_{22}(1 - \cos 2\theta)\} \\ &= \tfrac{1}{2}\{(g_{11} + g_{22}) + \cos 2\theta(g_{11} - g_{22}) + 2g_{12}\sin 2\theta\}. \end{aligned} \tag{5}$$

On setting $dF/d\theta$ equal to 0, one obtains

$$\theta = \tfrac{1}{2}\arctan\left(2g_{12}/(g_{11} - g_{22})\right). \tag{6}$$

If θ_0 is a solution to this equation, so is $\theta_0 \pm \pi/2$. As $F(\theta) = F(\theta + \pi)$, we may confine to the values of θ within the interval $[0, \pi)$. Thus, Eq. (6) always provides two values of θ; except for the case in which $F(\theta)$ is constant, F is maximum in correspondence to one of these two values and minimum in correspondence to the other. Stated differently, Eq. (6) associates with each point (x, y) in the space domain of the multi-image a pair of orthogonal directions: along one of them, f attains its maximum rate of change, along the other, its minimum.

Note. From a mathematical standpoint, the gradient of the multi-image f is tensor g. Indeed, f is a vector field, i.e., a vector valued function defined over a manifold (the x, y plane), hence its gradient must be a tensor.

Quite obviously, the whole theory applies whenever $m \geq 2$, i.e., when there is actually more than one image component. More details on this aspect are given in the Appendix, which also contains an elementary formulation of the maximization problem expressed by Eqs. (2) and (3).

In the remainder of this section, we shall consider the problem of deriving digital approximations for the gradient of a digital multi-image. More precisely, we shall derive digital approximations for θ_0 and $F(\theta_0)^{1/2}$, where θ_0 is the angle which determines the direction along which f has its maximum rate of change (in Euclidean metric), while $F(\theta_0)^{1/2}$ is the actual value of the maximum rate of change (also called "edge strength").

We shall make use of the surface fitting technique. Specifically, the given digital multi-image f will be approximated locally at each point by a linear multi-image $\hat{f}$ of same dimensionality m: $\hat{f}$ will then be used to compute approximated values of θ_0 and $F(\theta_0)^{1/2}$.

Let us first examine how $\hat{f}$ can be computed. We shall adopt for $\hat{f}$ the same notational conventions we have taken for f, namely $\hat{f} = (\hat{f}^1, \ldots, \hat{f}^m)$, $\hat{y} = \hat{f}(x) = (\hat{f}^1(x), \ldots, \hat{f}^m(x)) = (y^1, \ldots, y^m)$. Hence, for $j = 1, \ldots, m$, $\hat{y}^j = \hat{f}^j(x)$. As $\hat{f}$ is linear, for $j = 1, \ldots, m$, we have

$$\hat{y}^j = a_{j_1}x^1 + a_{j_2}x^2 + a_{j_3}. \tag{7}$$

So, in order to determine $\hat{f}$, we must get estimates of the coefficients a_{j_i}. This in turn can be done by means of well-known techniques. For example, the three coefficients in the right-hand side of Eq. (7) can be estimated by least-squares fitting the plane represented by that equation to the pixel values (in the jth image band) in a neighborhood of the given point (x^1, x^2). For example, we can use the pixel values $f^j(x^1, x^2)$, $f^j(x^1 + 1, x^2)$, $f^j(x^1, x^2 + 1)$, $f^j(x^1 + 1, x^2 + 1)$, as data points for a least squares fit of the plane represented by Eq. (7): the result is (see e.g., [13, Vol. 2, p. 107])

$$\begin{aligned} a_{j_1} &= \left(f^j(x^1 + 1, x^2) + f^j(x^1 - 1, x^2 + 1)\right)/2 \\ &\quad - \left(f^j(x^1, x^2) + f^j(x^1, x^2 + 1)\right)/2 \\ a_{j_2} &= \left(f^j(x^1, x^2 + 1) + f^j(x^1 + 1, x^2 + 1)\right)/2 \\ &\quad - \left(f^j(x^1, x^2) + f^j(x^1 + 1, x^2)\right)/2. \end{aligned}$$

The expression for a_{j_3} has not been reported: indeed, the coefficients a_{j_3}, $j = 1, \ldots, m$, are not actually needed. Notice that $\hat{f}(x)$ (or, more correctly, $\hat{V}_2 = \{\hat{f}(x) : x \in R^2\}$) can be considered an approximation to the tangent plane of V_2 at $y = f(x)$.

Once the relevant coefficients a_{j_i} have been computed, they can be used to obtain an estimate of θ_0 as follows. From Eq. (7) it follows that

$$a_{j_1} = \frac{\partial \hat{f}^j}{\partial x^1}, \qquad a_{j_2} = \frac{\partial \hat{f}^j}{\partial x^2}. \tag{8}$$

From these and Eq. (1) we can obtain approximations $\hat{g}_{hk}$ of the tensor coefficients g_{hk} as follows

$$\hat{g}_{11} = \sum_{j=1}^{m} a_{j_1}^2 \tag{9a}$$

$$\hat{g}_{22} = \sum_{j=1}^{m} a_{j_2}^2 \tag{9b}$$

$$\hat{g}_{12} = \sum_{j=1}^{m} a_{j_1} a_{j_2}. \tag{9c}$$

Then we are able to obtain an estimate $\hat{\theta}_0$ of θ_0 by means of Eq. (6). By substituting $\hat{\theta}_0$ into the right-hand side of Eq. (4), and then taking the positive square root, we then obtain an estimate of the edge strength locally at the point $x = (x^1, x^2)$ under consideration.

APPLICATION TO COLOR IMAGES

We shall discuss the above results in the particular case of color images ($m = 3$). A continuous color image can be regarded as a function mapping R^2 into the RGB space. Following the common notation, we shall write x, y instead of x^1, x^2. The color components will be denoted $R(x, y)$, $G(x, y)$ and $B(x, y)$, so that the image as a whole can be denoted $f = (R, G, B)$ or, more explicitly, $f(x, y) = (R(x, y), G(x, y), B(x, y))$. We shall write $\mathbf{r}, \mathbf{g}, \mathbf{b}$ for the unitary vectors associated with the R, G, and B axes, respectively. Then the vectors f_h, $h = 1, 2$ can be rewritten

$$\mathbf{u} = \frac{\partial R}{\partial x}\mathbf{r} + \frac{\partial G}{\partial x}\mathbf{g} + \frac{\partial B}{\partial x}\mathbf{b} \tag{10a}$$

$$\mathbf{v} = \frac{\partial R}{\partial y}\mathbf{r} + \frac{\partial G}{\partial y}\mathbf{g} + \frac{\partial B}{\partial y}\mathbf{b} \tag{10b}$$

Both $\mathbf{u}$ and $\mathbf{v}$ are functions of the two space coordinates x, y. The explicit formula for the angle θ can be rewritten

$$\theta = \tfrac{1}{2} \arctan\left(2g_{xy}/(g_{xx} - g_{yy})\right) \tag{11}$$

where

$$g_{xx} = \mathbf{u} \times \mathbf{u} = \left|\frac{\partial R}{\partial x}\right|^2 + \left|\frac{\partial G}{\partial x}\right|^2 + \left|\frac{\partial B}{\partial x}\right|^2 \tag{12a}$$

$$g_{yy} = \mathbf{v} \times \mathbf{v} = \left|\frac{\partial R}{\partial y}\right|^2 + \left|\frac{\partial G}{\partial y}\right|^2 + \left|\frac{\partial B}{\partial y}\right|^2 \tag{12b}$$

$$g_{xy} = \mathbf{u} \times \mathbf{v} = \frac{\partial R}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial G}{\partial x}\frac{\partial G}{\partial y} + \frac{\partial B}{\partial x}\frac{\partial B}{\partial y}. \tag{12c}$$

Quite obviously, the tensor components g_{xx}, g_{yy}, and g_{xy} are also functions of the space coordinates x, y.

To help understand the meaning of the above formulas, a few special cases will be discussed in detail.

Case 1. Assume that, we have at some point (x, y) in the space domain

$$\frac{\partial R}{\partial y} = \frac{\partial G}{\partial y} = \frac{\partial B}{\partial y} = 0$$

$$\frac{\partial B}{\partial x} = 0$$

$$\frac{\partial R}{\partial x} = -\frac{\partial G}{\partial x}.$$

In words, B is constant, no image components vary along the y axis, and R and G vary along x with the same absolute rate of change and opposite signs. In this case, $\mathbf{v} = 0$, hence $g_{yy} = g_{xy} = 0$, and we have

$$g_{xx} = 2\left|\frac{\partial R}{\partial x}\right|^2.$$

From Eq. 11 we have $\theta = 0$ or $\theta = \pi/2$. It is easily seen that the first solution corresponds to the maximum rate of change and the second to the minimum. So we have a maximum rate of change along x and the edge strength is $\sqrt{2}\,|\partial R/\partial x|$. Notice that evaluating a gradient for the whole multi-image as the vector sum of the gradients of the separate image components would provide a result of 0 in this case. On the contrary, the RMS approach would provide the correct results owing to the fact that the gradients of $R(x, y)$ and $G(x, y)$ have the same direction.

Case 2. Let us consider the case $\mathbf{u} = \mathbf{v}$. In this case, the rate of change along x and that along y are equal. From Eqs. (12a) through (12c), it follows that $g_{xx} = g_{yy} = g_{xy}$, hence, Eq. (11) provides either $\theta = \pi/4$ or $\theta = -\pi/4$. The maximum rate of change is found along $\theta = \pi/4$ and its value turns out to be

$$F\left(\frac{\pi}{4}\right)^{1/2} = \left\{\frac{1}{2}(g_{xx} + g_{yy}) + g_{xy}\sin\frac{\pi}{2}\right\}^{1/2} = \{2g_{xx}\}^{1/2}.$$

The RMS approach would provide the same result, again due to the fact that the gradients of the separate image bands all have the same direction (the one determined by $\theta = \pi/4$). It can be verified that the orthogonal solution $\theta = -\pi/4$ corresponds to a minimum of $F(\theta)$.

Case 3. Let us assume that, at some given point (x, y) in the space domain, the following situation holds

$$\frac{\partial R}{\partial x} = \frac{\sqrt{3}}{2}, \qquad \frac{\partial R}{\partial y} = \frac{1}{2}$$

$$\frac{\partial G}{\partial x} = -\frac{1}{2}, \qquad \frac{\partial G}{\partial y} = \frac{\sqrt{3}}{2}$$

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial y} = 0.$$

So, B is constant, while the gradients of R and G are orthogonal to one another and equal in magnitude. We have $g_{xx} = g_{yy} = 1$, $g_{xy} = 0$, hence Eq. (2) gives $\theta = 0$ or $\theta = \pi/2$. In this case, however, these values of θ are irrelevant as $F(\theta)$ is constant: $F(\theta) = 1$ for any θ. The rate of change turns out to be 1 (along any direction θ). Notice that the RMS approach would provide a value of $\sqrt{2}$.

Case 4. Let us consider a situation slightly different from that of Case 3, namely

$$\frac{\partial R}{\partial x} = \sqrt{3}, \qquad \frac{\partial R}{\partial y} = 1$$

$$\frac{\partial G}{\partial x} = -\frac{1}{2}, \qquad \frac{\partial G}{\partial y} = \frac{\sqrt{3}}{2}$$

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial y} = 0.$$

Hence B is again constant, and the gradients of R and G are again orthogonal to one another, however, they are not equal in magnitude. Here we have $g_{xx} = 13/4$, $g_{yy} = 7/4$, $g_{xy} = 3\sqrt{3}/4$. Equation (11) provides $\theta = 30°$, $\theta = 120°$. The maximum rate of change is found along $\theta = 30°$ and its value turns out to be 1.72. The RMS approach would provide a value of 2.23.

APPENDIX

In this Appendix, a simple derivation of the constrained maximum problem expressed by Eqs. (2) and (3) is provided. Let $F(\theta) = |f(x') - f(x)|^2$, where $x = (x^1, x^2)$ and $x' = (x^1 + \varepsilon\cos\theta, x^2 + \varepsilon\sin\theta)$. In words, $F(\theta)$ is the square of the length of the vector $f(x') - f(x)$ which represents the change of f for a small displacement from $x = (x^1, x^2)$ to a generic point x' on the circle of radius ε centered at x. We have

$$\begin{aligned} F(\theta) &= \sum_{i=1}^{m} \left[f^i(x^1, x^2) - f^i(x^1 + \varepsilon\cos\theta, x^2 + \varepsilon\sin\theta)\right]^2 \\ &= \sum_{i=1}^{m} \left[\frac{\partial f^i}{\partial x^1}\varepsilon\cos\theta + \frac{\partial f^i}{\partial x^2}\varepsilon\sin\theta\right]^2 \\ &= \varepsilon^2\left\{\left[\sum_{i=1}^{m}\left|\frac{\partial f^i}{\partial x^1}\right|^2\right]\cos^2\theta + \left[\sum_{i=1}^{m}\left|\frac{\partial f^i}{\partial x^2}\right|^2\right]\sin^2\theta \right. \\ &\qquad \left. + 2\left[\sum_{i=1}^{m}\frac{\partial f^i}{\partial x^1}\frac{\partial f^i}{\partial x^2}\right]\sin\theta\cos\theta\right\}. \end{aligned}$$

For g_{11}, g_{22}, g_{12} given by Eq. (1), we eventually get Eq. (4).

ACKNOWLEDGMENTS

This work has been developed during an assignment at the IBM Palo Alto Scientific Center. The author wishes to thank Drs. Ralph Bernstein, Jitendra V. Dave, Augustin Dubrulle, Alan H. Karp, Harwood Kolsky, and Piero Sguazzero for their valuable suggestions at various stages of this undertaking.

REFERENCES

1. M. Brady, Computational approaches to image understanding, *Comput. Rev.* **14**, No. 1, 1982, 3–71.
2. L. S. Davis, A survey of edge detection techniques, *Comput. Graphics Image Process.* **4**, 1975, 248–270.
3. S. Di Zenzo, Advances in Image Segmentation, *Image Vision Comput.* **1**, No. 4, 1983, 196–210.
4. K. S. Fu and J. K. Mui, A survey on image segmentation, *Pattern Recognit.* **13**, 1981, 3–16.
5. L. S. Kovasznay and H. M. Joseph, Processing of two-dimensional patterns by scanning techniques, *Science (Washington, D.C.)* **118**, 1953, 475–477.
6. L. S. Kovasznay and H. M. Joseph, Image Processing, in *Proc. IRE* **43**, 1955, 560–570.
7. R. Machuca and K. Phillips, Applications of vector fields to image processing, *IEEE Trans. Pattern Anal. Mach. Intell.* **5**, 1983, 316–329.
8. T. Peli and D. Malah, A study of edge detection algorithms, *Comput. Graphics Image Process.* **20**, 1982, 1–21.
9. J. M. Prewitt, Object enhancement and extraction, in *Picture Processing and Psychopictorics*, (B. S. Lipkin and A. Rosenfeld, Eds.), pp. 75–149, Academic Press, New York, 1970.
10. E. M. Riseman and M. A. Arbib, Computational techniques in the visual segmentation of static scenes, *Comput. Graphics Image Process.* **6**, 1977, 221–276.

11. L. G. Roberts, Machine perception of three-dimensional solids, in *Optical and Electrooptical Information Processing* (J. T. Tippett *et al.*, Eds), pp. 159–197, MIT Press, Cambridge, Mass., 1965.
12. A. Rosenfeld and M. Thurston, Edge and curve detection for visual scene analysis, *IEEE Trans. Comput.* **21**, 1972, 562–569.
13. A. Rosenfeld, M. Thurston, and Y. H. Lee, Edge and curve detection: Further experiments, *IEEE Trans. Comput.* **21**, 1972, 667–715.
14. A. Rosenfeld and A. C. Kak, *Digital Picture Processing*, Academic Press, New York, 1982.
15. K. S. Shanmugam, F. M. Dickey, and J. A. Green, An optimal frequency domain filter for edge detection in digital pictures, *IEEE Trans. Pattern Anal. Mach. Intell.* **PAMI-1**, No. 1, 1979 37–49.
16. B. J. Schachter, A. Lev, S. W. Zucker, and A. Rosenfeld, An application of relaxation methods to edge reinforcement, *IEEE Trans. Systems Man Cybern.* **SMC-7**, 1977, 813–816.
17. D. C. Morgenthaler and A. Rosenfeld, Multidimensional edge detection by hypersurface fitting, *IEEE Trans. Pattern Anal. Mach. Intell.* **PAMI-3**, 1981, 482–486.
18. R. Nevatia, A color edge detector and its use in scene segmentation, *IEEE Trans. Systems Man Cybern.* **SMC-7**, 1977, 820–826.
19. G. S. Robinson, Color edge detection, *Optical Engineering* **16**, 1977, 479–484.
20. P. V. Sankar, Color edge detection: A comparative study, University of Maryland TR-666 DAAG53-76C-0138, 1978.
21. S. W. Zucker and R. A. Hummel, A three-dimensional edge operator, *IEEE Trans. Pattern Anal. Mach. Intell.* **PAMI-3**, 1981, 324–331.