A Discrete Version of Green's Theorem

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Abstract—We formulate a discrete version of Green's theorem such that a summation of a two-dimensional function over a discrete region can be evaluated by the use of a summation over its discrete boundary. In many cases, the discrete Green theorem can result in computational gain. Applications of the discrete Green theorem to several typical image processing problems are demonstrated. We also apply it to analyze shapes of particle aggregates of Fe_2O_3. Experimental results of the shape study are presented.

Index Terms—Computer graphics, digital image processing, discrete geometry.

I. Introduction

GREEN'S theorem [1] in the continuous x-y plane can be written as

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy = \oint_C (f\, dx + g\, dy) \tag{1}$$

where C is the entire boundary of region R on the x-y plane. If we read (1) from left to right, we would say that (1) tells us how to evaluate a surface integral using a contour integral. In particular, if $f = 0$, (1) becomes

$$\iint_R \frac{\partial g}{\partial x}\, dx\, dy = \oint_C g\, dy. \tag{2}$$

Let $(\partial g/\partial x) = g'$; then (2) becomes

$$\iint_R g'\, dx\, dy = \int_c g\, dy. \tag{3}$$

In this paper we will first find a discrete version of (3), and then generalize it. The discrete Green theorem will tell us how to evaluate a summation of a two-dimensional function over a discrete region by the use of a summation over its boundary points. Since the number of boundary points usually is less than the number of points in the region enclosed by the boundary, the discrete Green theorem also suggests some computational advantages. In the discrete space, the boundary points of a region can be concisely represented by a Freeman chain code [2]. The discrete Green theorem can be used to solve some typical image processing problems such as: to calculate the moments of a region whose boundary is given; to determine whether a point is in a region whose boundary is given in a Freeman chain code; and to determine the area of intersection of two regions whose boundaries are given in a Freeman chain code.

In Section II, the discrete Green theorem corresponding to (3) will be described and proved. In Section III, we show some applications of the discrete Green theorem. In Section IV, generalization of the results in Section II is shown. In Section V, we show how to apply the discrete Green theorem to the shape study of particle aggregates. Section VI is the final discussion and conclusion.

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II. Describing the Discrete Version of Green's Theorem

A discrete space S is a set of lattice points $\{(h, k) | h, k$ are integers$\}$. A region R is a subset of S. A region R is finite if the number of lattice points of R is finite. Unless otherwise specified, we assume regions are finite in this paper. A region R is 8-connected if there is an 8-path between any two lattice points of R [3]. An interior point of R is a point of R all of whose four neighbors [4] are points of R. A boundary point of R is a point which is not an interior point, that is, at least one of its four neighbors is in $\bar{R}$ (the complement of R). An 8-connected region has no holes if all of its boundary points are 8-connected. In the following we will assume all regions are 8-connected and have no holes, unless specified. We can trace the boundary of a region by starting at an arbitrary point (x_0, y_0) and move along the boundary so that the region is always on the left-hand side. The sequence of boundary points visited by this tracing method is called the sequential boundary points. We can represent the sequential boundary points by the Freeman chain code $((x_0, y_0), a_0a_1a_2 \cdots a_{l-1})$. The elements of a Freeman chain code are defined in Fig. 1. A boundary point may be visited more than once using the aforesaid tracing method. They will be referred to as double points. The Freeman chain code of the boundary points of a region which contains only a single point is undefined. We exclude such a case in the sequel.

For each point $(h, k) \in S$ we can define a continuous half line $\{(x, y) | y = k, x > h\}$, which is referred to as 1-half line (1HL) of (h, k). Similarly $\{(x, y) | y = k, x < h\}$ is the 5-half line (5HL), $\{(x, y) | x = h, y > k\}$ is the 3-half line (3HL) and $\{(x, y) | x = h, y < k\}$ is the 7-half line (7HL). A boundary point p of region R is said to be a left point if there is a $p' \in \bar{R}$ such that p is on the 1HL of p' and no other point on both R and the 1HL of p' is closer to p'. Similarly, p is a right-point if there is a $p' \in \bar{R}$ such that p is on the 5HL of p' and no other point on both R and the 5HL of p' is closer to p'; p is a top point if there is a $p' \in \bar{R}$ such that p is on the 7HL of p'

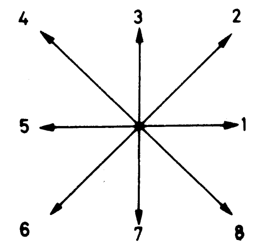

Fig. 1. Elements of Freeman chain code.

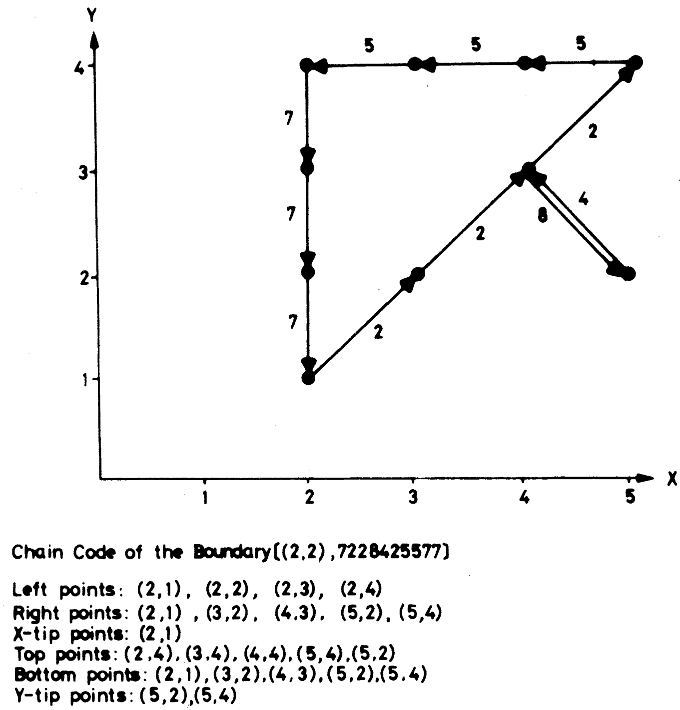

Fig. 2. An example to show some of the definitions in Section II.

and no other point on both R and the 7HL of p' is closer to p'; and p is a bottom point if there is a $p' \in \bar{R}$ such that p is on the 3HL of p' and no other point on both R and the 3HL of p' is closer to p'. A point is said to be an X-tip point if it is both a right point and a left point and not a double point. A point is said to be a Y-tip point if it is both a top point and a bottom point and not a double point. Fig. 2 shows an example to demonstrate these definitions.

Given a boundary point (x_i, y_i) and the Freeman chain elements $a_{i-1}, a_i (a_{i-1} = a_{l-1}$ if $i = 0)$ which sandwich (x_i, y_i), we can determine if (x_i, y_i) is a right point or a left point. But in general, we cannot determine whether (x_i, y_i) is an X-tip point by inspecting (a_{i-1}, a_i) alone. For certain (a_{i-1}, a_i)'s we can say definitely that (x_i, y_i) is not an X-tip point. For example, if $a_{i-1} = 2$ and $a_i = 4$, then we have a geometry as shown in Fig. 3. Since when we move from P_6 to P_4, the region should be on the left-hand side, so we can conclude that P_7, P_8, P_1, P_2, P_3 are not in region R (otherwise the chain code is not a_{i-1}, a_i). (x_i, y_i) is on the 5HL of P_1. So (x_i, y_i) is a right point. If P_5 is in R, then (x_i, y_i) is not an X-tip point. If P_5 is not in R, then (x_i, y_i) is a double point and not an X-tip point. Summarily, what we can say about (x_i, y_i) is that it is a right point and not an X-tip point. But we cannot say whether

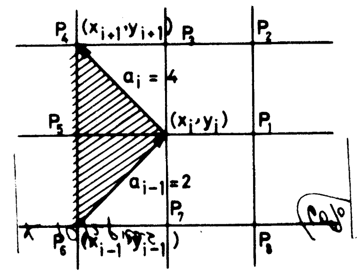

Fig. 3. Geometry for the case $a_{i-1} = 2$ and $a_i = 4$.

TABLE I

$D_Y(a_{i-1}, a_i) = 1$ INDICATES THAT THE iTH POINT IS A RIGHT AND NON-X-TIP POINT. $D_Y(a_{i-1}, a_i) = -1$ INDICATES THAT THE iTH POINT IS A LEFT AND NON-X-TIP POINT.

a_{i-1} \ a_i	1	2	3	4	5	6	7	8
1	0	1	1	1	1	0	0	0
2	0	1	1	1	1	0	0	0
3	0	1	1	1	1	0	0	0
4	0	1	1	1	1	0	0	0
5	-1	0	0	0	0	-1	-1	-1
6	-1	0	0	0	0	-1	-1	-1
7	-1	0	0	0	0	-1	-1	-1
8	-1	0	0	0	0	-1	-1	-1

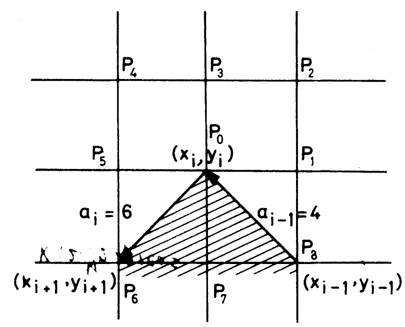

Fig. 4. Geometry for the case $a_{i-1} = 4$ and $a_i = 6$.

(x_i, y_i) is a left point or not. Following the similar argument for all possible a_{i-1}, a_i, we tabulate the results in Table I $(D_Y(a_{i-1}, a_i))$. $|D_Y(a_{i-1}, a_i)| = 1$ indicates that (x_i, y_i) is not an X-tip point. $D_Y(a_{i-1}, a_i) = -1$ indicates (x_i, y_i) is a left point and $D_Y(a_{i-1}, a_i) = +1$ indicates (x_i, y_i) is a right point. Consider another case when $a_{i-1} = 4$ and $a_i = 6$. The geometry is shown in Fig. 4. Moving from P_8 to P_6, we know that region R is below $P_8 P_0 P_6$, and P_1, P_2, P_3, P_4, P_5 are not in $\bar{R}$. P_0 is a right point and a left point. If P_7 is in R, then P_0 is an X-tip point; otherwise it is not. Hence, it is uncertain whether P_7 is an X-tip point or not. But it is certain to say that P_7 is a left point. Using the similar argument we can establish $C_Y(a_{i-1}, a_i)$ (see Table II) such that $C_Y(a_{i-1}, a_i) = 1$ indicates that (x_i, y_i) is a left point.

Let $f(m, n)$ be a function defined over all the lattice points and R is a region in $\{(h, k) | h \geqslant 0, k \geqslant 0$, and h, k are integers$\}$ and R is 8-connected without holes. R has at least two points. The boundary of R is represented by Freeman chain code $((x_0, y_0), a_0 a_1 \cdots a_{l-1})$.

TABLE II
$C_Y(a_{i-1}, a_i)$. $C_Y(a_{i-1}, a_i) = 1$ INDICATES THE ith POINT IS A LEFT POINT

a_{i-1} \ a_i	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	0	0
3	0	0	0	0	0	1	1	0
4	0	0	0	0	0	1	1	1
5	1	0	0	0	0	1	1	1
6	1	1	0	0	0	1	1	1
7	1	1	1	0	0	1	1	1
8	1	1	1	1	0	1	1	1

TABLE III
DEFINITION OF a_{ix} AND a_{iy} FOR EACH FREEMAN CODE ELEMENT a_i

a_i	a_{ix}	a_{iy}
1	1	0
2	1	1
3	0	1
4	-1	1
5	-1	0
6	-1	-1
7	0	-1
8	1	-1

We define $F_x(m, n) = \sum_{i=0}^{m} f(i, n)$,

$$x_{i+1} = x_i + a_{ix}$$
$$y_{i+1} = y_i + a_{iy} \tag{4}$$

where a_{ix}, a_{iy} are defined in Table III. The sequential boundary point $(x_0, y_0), (x_1, y_1) \cdots (x_{l-1}, y_{l-1})$ is denoted by B.

Lemma 1: If $(x_i, y_i) \in B$ and is a right and non-X-tip point, then there is a $(x_j, y_j) \in B$ such that $x_j \leqslant x_i$, $y_j = y_i$, and (x_j, y_j) is a left and non-X-tip point. All the lattice points between (x_j, y_j) and (x_i, y_i) are in R.

Proof: Since (x_i, y_i) is a right point, there is a $P \in \bar{R}$ such that (x_i, y_i) is on 5HL $\vec{P}_0$ of P. See Fig. 5. Let (x_t, y_i) be a point in R such that $\{(x_t, y_i), (x_t + 1, y_i), \cdots, (x_i - 1, y_i), (x_i, y_i)\} \in R \cap \vec{P}_0$ and $(x_t - 1, y_i) \in \bar{R}$. If $x_t = x_i$, then (x_i, y_i) is a double point and $(x_j, y_j) = (x_i, y_i)$. If $x_t \neq x_i$, then (x_t, y_i) is a left point since $(x_t - 1, y_i) \in \bar{R}$ and is not a right point since $(x_t, y_i) \in R$. Hence, (x_t, y_i) is a left point and not an X-tip point and $(x_j, y_j) = (x_t, y_i)$.

For each non-X-tip right point P_r, we can define a set of lattice points (called HLS) $P_l(P_r) = \{(h, k) | P_l$ is the non-X-tip left point corresponding to P_r (using Lemma 1), h, k are integers, and (h, k) lies on the horizontal line segment $\overline{P_l P_r}\}$. Let $\bar{H}$ be all HLS's defined by all the non-X-tip right points of B. $\bar{T}$ is the set of all X-tip points of B. See Fig. 6 for an example. Then $R = \bar{H} \cup \bar{T}$ $(\bar{H} \cap \bar{T} = \phi)$. We have

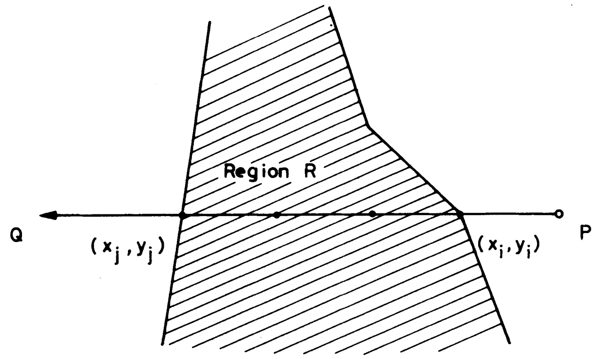

Fig. 5. Geometry for Lemma 1.

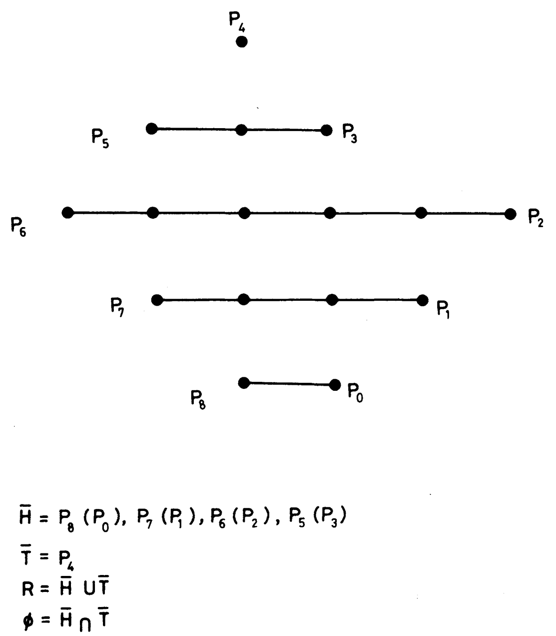

Fig. 6. An example to demonstrate the proof of Lemma 1.

$$\begin{aligned} \sum_{(m,n)\in R} f(m,n) &= \sum_{(m,n)\in \bar{H}} f(m,n) + \sum_{(m,n)\in \bar{T}} f(m,n) \\ &= \sum_{h\in \bar{H}} \sum_{(m,n)\in h} f(m,n) + \sum_{(m,n)\in \bar{T}} f(m,n) \end{aligned} \tag{5}$$

where h is any HLS of $\bar{H}$.

Letting the endpoints of $h \in \bar{H}$ be (m_2, n) and (m_1, n), (5) becomes

$$\begin{aligned} &\sum_{h\in \bar{H}} \left(\sum_{i=m_2}^{m_1} f(i,n) \right) + \sum_{(m,n)\in \bar{T}} f(m,n) \\ &= \sum_{h\in \bar{H}} \left[\sum_{i=0}^{m_1} f(i,n) - \sum_{i=0}^{m_2} f(i,n) + f(m_2, n) \right] \\ &\quad + \sum_{(m,n)\in \bar{T}} f(m,n) \end{aligned} \tag{6}$$

$$= \sum_{i=0}^{l-1} F_x(x_i, y_i) D_Y(a_{i-1}, a_i) + \sum_{\substack{(m,n) \text{ are} \\ \text{non-}X\text{-tip} \\ \text{and left} \\ \text{point}}} f(m,n) + \sum_{(m,n)\in \bar{T}} f(m,n) \tag{7}$$

$$= \sum_{i=0}^{l-1} F_x(x_i, y_i) D_Y(a_{i-1}, a_i) + \sum_{i=0}^{l-1} f(x_i, y_i) C_Y(a_{i-1}, a_i). \tag{8}$$

Summarizing the aforesaid results, we have Theorem 1.

Theorem 1: Let R be a discrete 8-connected region without holes in the subspace $S' = \{(h, k) | h \geqslant 0, k \geqslant 0, h, k \text{ are integers}\}$. R contains more than one lattice point. The sequential boundary of R is B represented by a Freeman chain code as $((x_0, y_0), a_0 a_1 \cdots a_{l-1})$ such that region R is seen on the left-hand side as one moves along the sequential boundary points. $f(m, n)$ is a function defined over S'. The following equation holds:

$$\sum_{(m,n)\in R} f(m,n) = \sum_{i=0}^{l-1} [F_x(x_i, y_i) D_Y(a_{i-1}, a_i) + f(x_i, y_i) C_Y(a_{i-1}, a_i)] \tag{9}$$

where

$$F_x(m,n) = \sum_{i=0}^{m} f(i,n)$$

$$x_{i+1} = x_i + a_{iX}$$

$$y_{i+1} = y_i + a_{iY}.$$

III. Applications

In this section, we will examine various applications of Theorem 1.

1) Calculating the Moments: The u, v moment of a region R is defined as

$$M_{uv} = \sum_{(m,n)\in R} m^u n^v \tag{10}$$

where (m, n) is the coordinate of any point in R. If R is in $S' = \{(h, k) | h \geqslant 0, k \geqslant 0, h, k \text{ are integers}\}$ and R has no holes, by the use of (9), (10) becomes

$$M_{uv} = \sum_{i=0}^{l-1} [F_x(x_i, y_i) D_Y(a_{i-1}, a_i) + f(x_i, y_i) C_Y(a_{i-1}, a_i)] \tag{11}$$

where (x_i, y_i)'s are the sequential boundary points. Equation (11) effectively tells us how to calculate the u, v moments of a region R if only the sequential boundary points are given. The function F_x in (11) is $F_x(m, n) = \sum_{i=0}^{m} i^u n^v$. For the first and second moments, a closed form for $F_x(m, n)$ can be easily found. Thus we have the following.

i) $u = 0, v = 0$: We have

$$F_x(m,n) = \sum_{i=0}^{m} i^0 n^0 = \sum_{i=0}^{m} 1 = m$$

and

$$M_{00} = \sum_{i=0}^{l-1} [x_i D_Y(a_{i-1}, a_i) + C_Y(a_{i-1}, a_i)].$$

M_{00} is the number of discrete lattice points in R or the area of R. In [2], Freeman suggested a formula to calculate the area which is defined as the number of cells enclosed by the boundary, not the number of lattice points in the region. Therefore, our formula and Freeman's formula will not produce the same result for a given Freeman chain code.

ii) $u = 1, v = 0$: We have

$$F_x(m,n) = \sum_{i=0}^{m} i = m(m+1)/2$$

and

$$M_{10} = \sum_{i=0}^{l-1} [(x_i(x_i+1)/2) D_Y(a_{i-1}, a_i) + x_i C_y(a_{i-1}, a_i)].$$

iii) $u = 0, v = 1$: We have

$$F_x(m,n) = \sum_{i=0}^{m} n = mn$$

and

$$M_{01} = \sum_{i=0}^{l-1} x_i y_i D_Y(a_{i-1}, a_i) + y_i C_Y(a_{i-1}, a_i).$$

iv) $u = 1, v = 1$: We have

$$F_x(m,n) = \sum_{i=0}^{m} in = nm(m+1)/2$$

and

$$M_{11} = \sum_{i=0}^{l-1} [(x_i y_i (x_i + 1)/2) D_Y(a_{i-1}, a_i) + x_i y_i C_Y(a_{i-1}, a_i)].$$

v) $u = 2, v = 0$: We have

$$F_x(m,n) = \sum_{i=0}^{m} i^2 = m(m+1)(2m+1)/6$$

and

$$M_{20} = \sum_{i=0}^{l-1} [(x_i(x_i+1)(2x_i+1)/6) D_Y(a_{i-1}, a_i) + x_i^2 C_Y(a_{i-1}, a_i)].$$

vi) $u = 0, v = 2$: We have

$$F_x(m,n) = \sum_{i=0}^{m} n^2 = mn^2$$

and

$$M_{02} = \sum_{i=0}^{l-1} [x_i y_i^2 D_Y(a_{i-1}, a_i) + y_i^2 C_Y(a_{i-1}, a_i)].$$

Notice that if we use (10) to calculate the moments, the number of computations is proportional to M_{00} (i.e., the area) while if we use (11) to calculate the moments, the number of computations is proportional to the length of the sequential boundary which is linearly related to $\sqrt{M_{00}}$. This is a significant computational gain.

2) Determining Whether a Point is in a Region: Given a region R in S' by its sequential boundary $((x_0, y_0), a_0 a_1 \cdots a_{l-1})$ and a point $(h, k) \in S'$, we want to determine whether (h, k) is in R.

Define

$$J(m-h, n-k) = \begin{cases} 0 & \text{if } m \neq h \text{ or } n \neq k \\ 1 & \text{if } m = h, \quad n = k. \end{cases}$$

Then

$$\sum_{(m,n)\in R} J(m-h, n-k) = \begin{cases} 1 & \text{if } (h,k) \text{ is in } R. \\ 0 & \text{if } (h,k) \text{ is not in } R. \end{cases}$$

Applying Theorem 1, we have

$$\sum_{(m,n)\in R} J(m-h, n-k)$$
$$= \sum_{i=0}^{l-1} [\Delta_x(x_i - h, y_i - k) D_Y(a_{i-1}, a_i)$$
$$+ J(x_i - h, y_i - k) C_Y(a_{i-1}, a_i)]$$

where

$$\Delta_x(x_i - h, y_i - k) = \sum_{i=0}^{x_i} J(i - h, y_i - k)$$
$$= \begin{cases} 1 & \text{if } y_i = k \text{ and } x_i \geq h. \\ 0 & \text{otherwise.} \end{cases}$$

The number of computations is proportional to the length of the sequential boundary.

3) Determining the Area of the Intersection of Two Regions Defined by Their Sequential Boundary Points $((x_0, y_0), a_0 a_i \cdots a_{l-1})$ and $((x_0', y_0'), a_0' a_1' \cdots a_{l'-1}')$: Let R be the region corresponding to $((x_0, y_0), a_0 a_1 \cdots a_{l-1})$. Define $f(x, y) = \sum_{(m,n)\in R} J(m-x, n-y)$. Then we have $f(x, y) = 1$ if (x, y) is in R and $f(x, y) = 0$ if (x, y) is not in R. The area of the intersection between R and R' is

$$I(R, R') = \sum_{(x,y)\in R'} f(x,y)$$
$$= \sum_{(x,y)\in R'} \sum_{(m,n)\in R} J(m-x, n-y). \tag{12}$$

Applying Theorem 1 to (12) twice, we have

$$I(R,R') = \sum_{j=0}^{l'-1} \sum_{i=0}^{l-1} [A(x_i, y_i, x_j', y_j') D_Y(a_{i-1}, a_i)$$
$$\cdot D_Y(a_{j-1}', a_j')$$
$$+ B(x_i, y_i, x_j', y_j') C_Y(a_{i-1}, a_i) D_Y(a_{j-1}', a_j')$$
$$+ \Delta_x(x_i - x_j', y_i - y_j') D_Y(a_{i-1}, a_i) C_Y(a_{j-1}', a_j')$$
$$+ J(x_i - x_j', y_i - y_j') C_Y(a_{i-1}, a_i) C_Y(a_{j-1}', a_j')] \tag{13}$$

where

$$A(x_i, y_i, x_j', y_j') = \sum_{P=0}^{x_j'} \Delta_x(x_i - P, y_i - y_j')$$
$$= \begin{cases} x_j' & \text{if } x_j' \leq x_i, \ y_j' = y_i \\ x_i & \text{if } x_j' > x_i, \ y_j' = y_i \\ 0 & \text{otherwise} \end{cases}$$

$$B(x_i, y_i, x_j', y_j') = \sum_{P=0}^{x_j'} J(x_i - P, y_i - y_j')$$
$$= \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } x_j' \geq x_i, \ y_j' = y_i \end{cases}$$

and (x_i, y_i), (x_j', y_j') are the ith and jth sequential boundary point on R and R', respectively.

The number of computations of using (13) is proportional to the product of the lengths of the boundary sequential points of R and R'. Other methods for area intersection and point inclusion can be seen in [5].

IV. Generalization

Following arguments similar to those in Section II we can establish $D_x(a_{i-1}, a_i)$ (see Table IV) and $C_x(a_{i-1}, a_i)$ (see Table V). Theorem 2, described below, can be proved in a manner similar to Theorem 1.

Theorem 2: Let R be a discrete 8-connected region without holes in the subspace $S' = \{(h, k) | h \geq 0, k \geq 0, h, k \text{ are integers}\}$. R contains more than one lattice point. The sequential boundary of R is B represented by Freeman chain code as $((x_0, y_0), a_0 a_1 \cdots a_{l-1})$ such that region R is seen on the left-hand side as one moves along the sequential boundary points. $g(m, n)$ is a function defined over S'. The following equation holds:

$$-\sum_{(m,n)\in R} g(m,n) = + \sum_{i=0}^{l-1} G_y(x_i, y_i) D_x(a_{i-1}, a_i)$$
$$- g(x_i, y_i) C_x(a_{i-1}, a_i)$$

TABLE IV
$D_x(a_{i-1}, a_i)$. $D_x(a_{i-1}, a_i) = +1$ INDICATES THAT THE ith POINT IS A BOTTOM AND NON-Y-POINT. $D_x(a_{i-1}, a_i) = -1$ INDICATES THAT THE ith POINT IS A TOP AND NON-Y-TIP-POINT.

a_{i-1} \ a_i	1	2	3	4	5	6	7	8
1	+1	1	1	0	0	0	0	1
2	1	1	1	0	0	0	0	1
3	0	0	0	-1	-1	-1	-1	0
4	0	0	0	-1	-1	-1	-1	0
5	0	0	0	-1	-1	-1	-1	0
6	0	0	0	-1	-1	-1	-1	0
7	1	1	1	0	0	0	0	0
8	1	1	1	0	0	0	0	0

where

$$G_Y(m,n) = \sum_{i=0}^{n} g(m,i)$$

$$x_{i+1} = x_i + a_{ix}$$

and

$$y_{i+1} = y_i + a_{iY}.$$

Combining Theorem 1 and Theorem 2, we obtain Theorem 3 which is the discrete version of Green's theorem.

Theorem 3 (Discrete Green's Theorem): Let R be a discrete 8-connected region without holes in the subspace $S' = \{(h, k) \mid h \geqslant 0, k \geqslant 0, h, k \text{ are integers}\}$. R contains more than one lattice point. The sequential boundary of R is B represented by a Freeman chain code as $((x_0, y_0), a_0 a_1 \cdots a_{l-1})$ such that R is seen on the left-hand side as one moves along the sequential boundary points. $f(m,n)$ and $g(m,n)$ are functions defined over S'. The following equation holds:

$$\sum_{(m,n)\in R} (f(m,n) - g(m,n))$$

$$= \sum_{i=0}^{l-1} \Big[F_x(x_i, y_i)\, D_Y(a_{i-1}, a_i) + G_Y(x_i, y_i)\, D_x(a_{i-1}, a_i)) + \sum_{i=0}^{l-1} (f(x_i, y_i)\, C_Y(a_{i-1}, a_i) - g(x_i, y_i)\, C_x(a_{i-1}, a_i) \Big] \tag{14}$$

where

$$F_x(m,n) = \sum_{j=0}^{m} f(j,n)$$

$$G_Y(m,n) = \sum_{j=0}^{n} g(m,j)$$

$$x_{i+1} = x_i + a_{ix}$$

TABLE V
$C_x(a_{i-1}, a_i)$. $C_x(a_{i-1}, a_i) = 1$ INDICATES THAT THE ith POINT IS A BOTTOM POINT.

a_{i-1} \ a_i	1	2	3	4	5	6	7	8
1	1	1	1	1	1	0	0	1
2	1	1	1	1	1	1	0	1
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	1
5	1	0	0	0	0	0	0	1
6	1	1	0	0	0	0	0	1
7	1	1	1	0	0	0	0	1
8	1	1	1	1	0	0	0	1

and

$$y_{i+1} = y_i + a_{iy}.$$

Define

$$\nabla_x F_x(m,n) = F_x(m,n) - F_x(m-1,n)$$

and

$$\nabla_Y G_Y(m,n) = G_Y(m,n) - G_Y(m,n-1).$$

Equation (14) can be written as

$$\sum_{(m,n)\in R} (\nabla_x F_x(m,n) - \nabla_Y G_Y(m,n))$$

$$= \sum_{i=0}^{l-1} [F_x(x_i,y_i)\, D_Y(a_{i-1}, a_i) + G_Y(x_i,y_i)\, D_x(a_{i-1}, a_i)] + \sum_{i=0}^{l-1} [\nabla_x F_x(x_i,y_i)\, C_Y(a_{i-1}, a_i) - \nabla_Y G_Y(x_i,y_i)\, C_x(a_{i-1}, a_i)] \tag{15}$$

because

$$\nabla_x F_x(m,n) = \sum_{i=0}^{m} f(i,n) - \sum_{i=0}^{m-1} f(i,n) = f(m,n)$$

and similarly

$$\nabla_Y G_Y(m,n) = g(m,n).$$

The similarity in appearance between (1) and (15) is apparent.

The discrete Green theorem also holds if region R contains holes. If R contains holes, then the boundary points are not 8-connected. Instead the boundary points consist of several 8-connected closed arcs. Assume that region R has $k-1$ holes, then the Freeman chain code representing the boundary of R is

$$[(x_{01}, y_{01}), a_{01}, a_{11}, \cdots, a_{(l_1-1)1}),$$
$$((x_{02}, y_{02}), a_{02}, a_{12}, \cdots, a_{(l_2-1)2}),$$
$$\cdots ((x_{0k}, y_{0k}), a_{0k} a_{1k} \cdots a_{(l_k-1)k}]$$

+ (1,2)

(0,1)+ + (2,1)

+ (1,0)

The region R.
Freeman chain code of the outer boundary ((0,1), 8246).

Freeman chain code of the inner boundary ((0,1), 2864)

Calculating the summation of the right hand side of (16) over the outer boundary is shown below:

i	a_{i-1}	a_i	$D_Y(a_{i-1},a_i)$	$C_Y(a_{i-1},a_i)$	X_i	Y_i	$x_i\, D_Y(a_{i-1},a_i) + C_Y(a_{i-1},a_i)$
0	6	8	-1	1	0	1	1
1	8	2	0	1	1	0	1
2	2	4	1	0	2	1	2
3	4	6	0	1	1	2	1

$$\sum_{i=0}^{3} x_i\, D_Y(a_{i-1},a_i) + C_Y(a_{i-1},a_i) = 5 \qquad (5\text{-}1)$$

Calculating the summation of the right hand side of (16) over the inner boundary is shown below:

i	a_{i-1}	a_i	$D_Y(a_{i-1},a_i)$	$C_Y(a_{i-1},a_i)$	X_i	Y_i	$x_i\, D_Y(a_{i-1},a_i) + C_Y(a_{i-1},a_i)$
0	4	2	1	0	0	1	0
1	2	8	0	0	1	2	0
2	8	6	-1	1	2	1	-1
3	6	4	0	0	1	0	0

$$\sum_{i=0}^{3} x_i\, D_Y(a_{i-1},a_i) + C_Y(a_{i-1},a_i) = -1 \qquad (5\text{-}2)$$

(5-1) + (5-2) = 4 = the area of R

Fig. 7. Demonstration of using formula (16) to calculate the area of region R which has a hole.

where

$$(x_{0i}, y_{0i}), \qquad i = 1, 2, \cdots, k$$

is the coordinate of an arbitrary point of the ith boundary arc. $(a_{0i}a_{1i} \cdots a_{(l_i-1)i})$ is the chain code obtained by starting at (x_{0i}, y_{0i}) and moving along the corresponding closed arc such that region R is always on the left-hand side. Formula (14) becomes

$$\sum_{(m,n)\in R} (f(m,n) - g(m,n))$$

$$= \sum_{j=1}^{k} \sum_{i=0}^{l_j-1} [F_x(x_{ij}, y_{ij})\, D_Y(a_{ij-1}, a_{ij}) + G_Y(x_{ij}, y_{ij})\, D_x(a_{ij-1}, a_{ij}) + f(x_{ij}, y_{ij})\, C_Y(a_{ij-1}, a_{ij}) - g(x_{ij}, y_{ij})\, C_x(a_{ij-1}, a_{ij})] \qquad (16)$$

where

$$x_{(i+1)j} = x_{ij} + a_{ijx}$$

$$y_{(i+1)j} = y_{ij} + a_{ijy}.$$

To prove (16) is straightforward, we will use a simple example to demonstrate (16) is true. Consider that a discrete region R_1 contains four points $R_1 = \{(0, 1), (1, 0), (2, 1), (1, 2)\}$. R_1 has one hole. The Freeman chain code of the outer boundary is $\{(0, 1), 8246\}$, and that of the inner boundary is $\{(0, 1), 2864\}$. Define $f(m, n) = 1, g(m, n) = 0$; then $\sum_{(m,n)\in R_1} f(m, n)$ is the area of R_1 which is apparently equal to four. Applying (16), we can obtain the same result. Fig. 7 shows the calculations of applying (16) to find the area.

It should be pointed out that the discrete Green theorem is not true if region R is not a subset of $S' = \{(h, k) | h \geqslant 0, k \geqslant 0, h, k \text{ are integers}\}$. To extend the discrete Green theorem such that region R may be in several different quadrants involves defining a summation method which reflects the "direction" of summation. Since we can represent most pictures as a subset of S', we will not make an extension of the discrete Green theorem such that region R can lie in different quadrants. But such extension is possible.

V. The Discrete Green Theorem as Applied to Particle Shape Analysis

The dynamics of the electrical mobility of an aggregate of particles is related to the shape of the aggregate. To understand the dynamics and the electrical mobility of particles is of vital importance in developing a control process such as to minimize the generation of harmful particles during coal liquefaction. We applied the digital image processing techniques in general and the discrete Green theorem in particular to study the shape effects.

Several microscope pictures of aggregates of Fe_2O_3 particles have been digitized. A COMTAL image processing system is used to process the digital images. Since the background is clear, the boundary points of each aggregate can be obtained easily be a simple thresholding technique. The boundary points are then encoded in a Freeman chain code. The boundary of an aggregate in Fig. 8 thus obtained is shown in Fig. 9. Then, using (10), we calculate $M_{00}, M_{10}, M_{01}, M_{11}, M_{20}, M_{02}$ of each particle aggregate. The centroid of the aggregate is defined as $(M_{10}/M_{00}, M_{01}/M_{00})$. The aspect ratio of the aggregate is defined as

$$A \text{ (aspect ratio)} = \frac{\max(\alpha, \beta)}{\min(\alpha, \beta)}$$

where α, β are eigenvalues to the matrix

$$\begin{bmatrix} C_{20} & C_{11} \\ C_{11} & C_{02} \end{bmatrix}$$

and

$$C_{20} = \frac{M_{20}}{M_{00}} - \left(\frac{M_{10}}{M_{00}}\right)^2$$

$$C_{02} = \frac{M_{02}}{M_{00}} - \left(\frac{M_{01}}{M_{00}}\right)^2, \qquad C_{11} = \frac{M_{11}}{M_{00}} - \left(\frac{M_{01}}{M_{00}} \frac{M_{10}}{M_{00}}\right).$$

We use the technique described in Section III-2 to determine whether the centroid is located in the aggregate. The result reported by the machine is also shown in Fig. 9. All the programs are written in Fortran in an interactive mode. The user uses the cursor to select a particle aggregate. The computer generates a report as shown in Fig. 9 almost instantly.

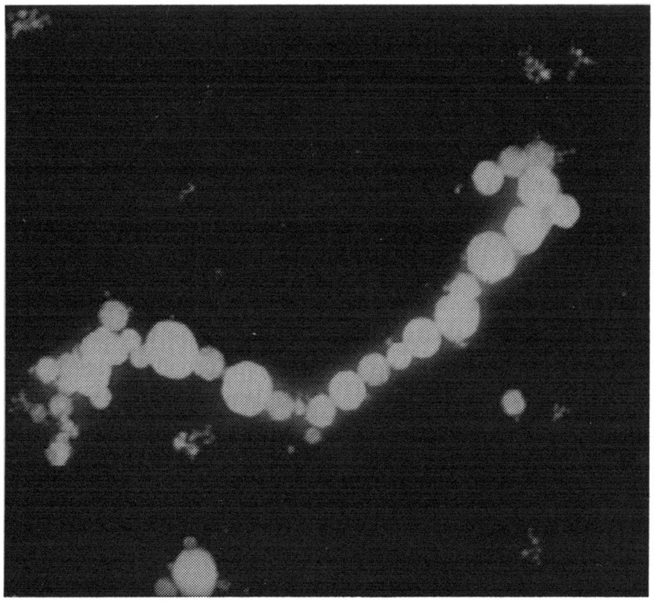

Fig. 8. One of the pictures of aggregates Fe_2O_3.

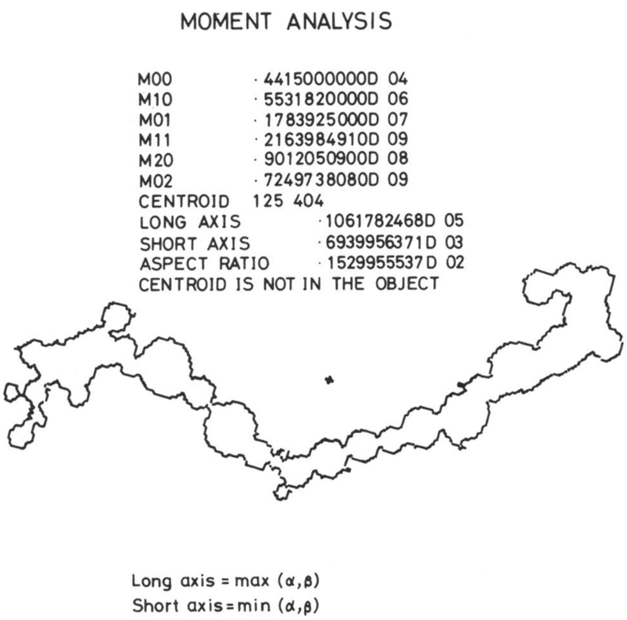

Fig. 9. An example of the contour and the moments reported by the image processor of the aggregate seen in Fig. 8.

VI. Conclusion and Discussion

We have formulated a discrete version of Green's theorem. The discrete Green theorem suggests a method to evaluate a summation of a point function $f(m, n)$ over a discrete region by a summation over the discrete boundary of that region. Since, in general, the number of discrete boundary points is less than the number of points in the discrete region enclosed by that boundary, the discrete Green theorem suggests a potential computational advantage for some point functions. Several applications of the discrete Green theorem to some typical image processing problems are demonstrated. They are: to calculate the moments of a region, given the Freeman chain code of the boundary; to determine whether a point is in a given region whose boundary points, in terms of the Freeman chain code, are known; and to determine the area of the intersection of two discrete regions whose boundary points are given in terms of the Freeman chain code. We also apply the discrete Green theorem to study the shape of particle aggregates. An experimental result of this shape study is shown.

The discrete Green theorem uses the Freeman chain code to represent the boundary points. Some results shown in [2] can also be obtained by the use of the discrete Green theorem. It will be a fair statement if we say that the discrete Green theorem is an extension and generalization of the previous work done by Freeman [2].

The discrete Green theorem holds when the discrete region is entirely in one quadrant. The discrete Green theorem fails if the discrete region lies in several quadrants. It is possible to remove this failure by defining a new way to do summation. We feel such effort is not necessary because most pictures can be represented as a subset of the first quadrant.

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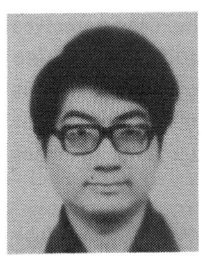

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